BTech (Mechanical) V Semester Course 2022-23

Fluid Mechanics -II (MEC3310)

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Introduction

- Contents
  - Introduction to Course
  - Role of Fluid Mechanics
  - Connection with EMEC2310 (Fluid Mechanics - I)

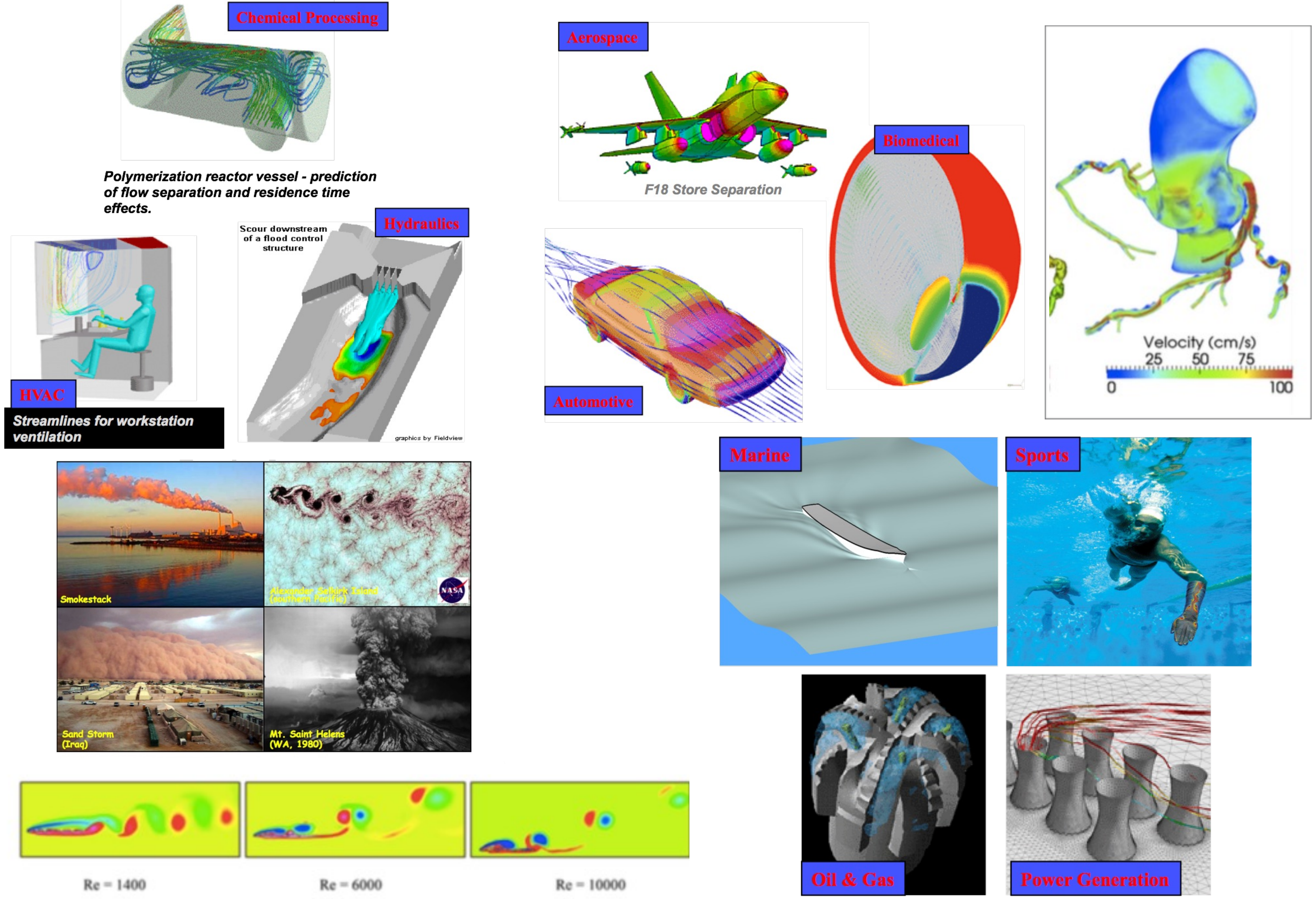
Introduction

Course Objectives

- To be able to model / express viscous fluid flow using physical principles.
- To develop an understanding of viscous flow behaviour by examining solutions to well known problems.
- Development of relevant mathematical skills of approximation and analytical solution for viscous fluid flow problems.
- Development of an understanding of important fluid flow phenomenon like formation / evolution of Boundary-Layer
- To be develop a basic understanding of Turbulence in fluid flows and the need for statistical approach to tackle Turbulence flows.

Role of Fluid Mechanics

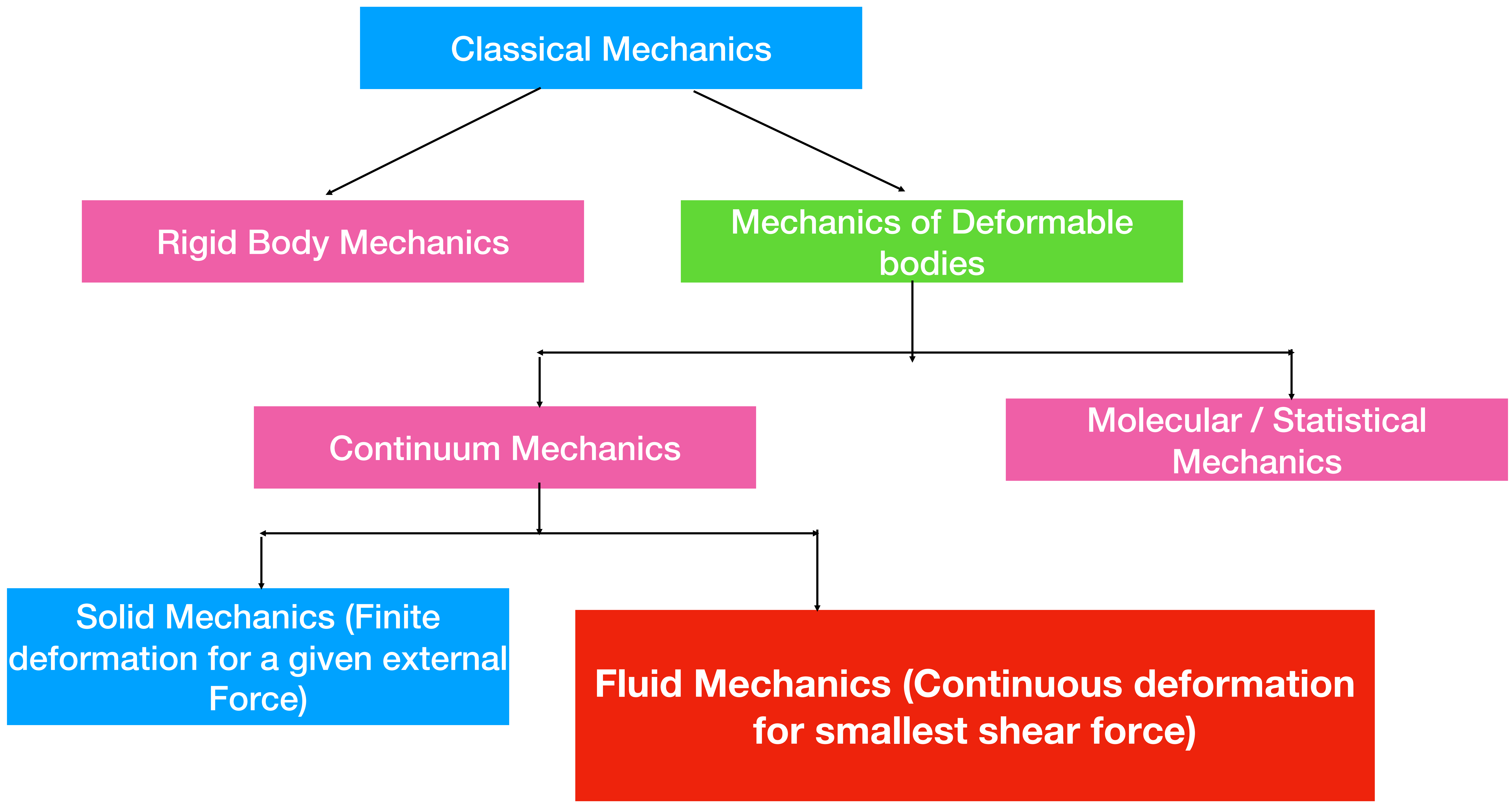
- **Technology / Process:** Power generation, Land / Air/ Sea transport systems / vehicles. Earth-fixed structures, heat transfer / removal , transport of fluids, sports etc
- **Environments/ Geophysics:** Atmospheric / Oceanic Flows, weather patterns, hurricanes / tornadoes, pollution dispersion, convection in earth's core
- **Biological Systems:** Respiratory and Blood Circulation, Fluid flow in Brain
- **Astrophysical Systems:** Stellar Convection, Supernovae explosions, astrophysical jets



Flow of lubricating mud over drill bit

Flow around cooling towers

Mechanics and Fluid Mechanics



Fluid Mechanics I : Recap

- Fluid as a continuum (macroscopic viewpoint), notion of a fluid particle)
- Surface and Body forces: the state of stress at a point, body force intensity at a point
- Fluid Statics: fluid pressure, hydrostatic equation and related application
- Kinematics
  - Eulerian and Lagrangian viewpoint
  - Flow visualisation: Stream lines, Streak lines, path lines, Material lines.
  - Material Derivative of any property
- Dynamics of inviscid flows: Euler and Bernoulli's Eqn
- Finite System and Control Volume approach: Reynolds Transport Eqn
- Viscous Pipe Flow and Energy Loss

Fluid Mechanics - II (Extension of Knowledge)

- Module 1: Mathematical Model of Dynamics Viscous Flows
  - Kinematics: Strain rates at a point, Vorticity, Velocity gradient System
  - Governing Equations
  - Dimensionless Formulations and Dynamical Similarity
  - Exact Solutions
- Module 2: Boundary Layer Theory
  - Boundary Layer Equations, Boundary -Layer Characteristics
  - Flow Separation and Its Control
  - Approximate Integral Method
- Module 3: Turbulent Flow
  - Characteristics of Turbulent Flows, Length and Time Scales
  - Need for Statistical approach
  - Mean flow equations and closure problem

Texts and supplementary study material

- Lecture notes
- Viscous Fluid Flow by FM White,Mc GrawHill, 3rd Edition
- Fluid Mechanics, 4E,Pijush K. Kundu and Ira M. Cohen, Academic Press 2008
- Incompressible Flow, 3rd Edition, Ronald L Panton, Wiley
- Advanced Fluid Mechanics, Som and Biswas, Narosa Publication
- **Supplementary Materials (to be downloaded from my webpage)**
  - **Algebra and Calculas of Vector and Tensors**
  - **Important Results and theorems of vector calculus**

Thanks

BTech (Mechanical) V Semester Course 2022-23

Fluid Mechanics -II (MEC3310)

Module 1: Basics of Viscous Flows

Lecture 2: Local Deformation and Rotation

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Objectives

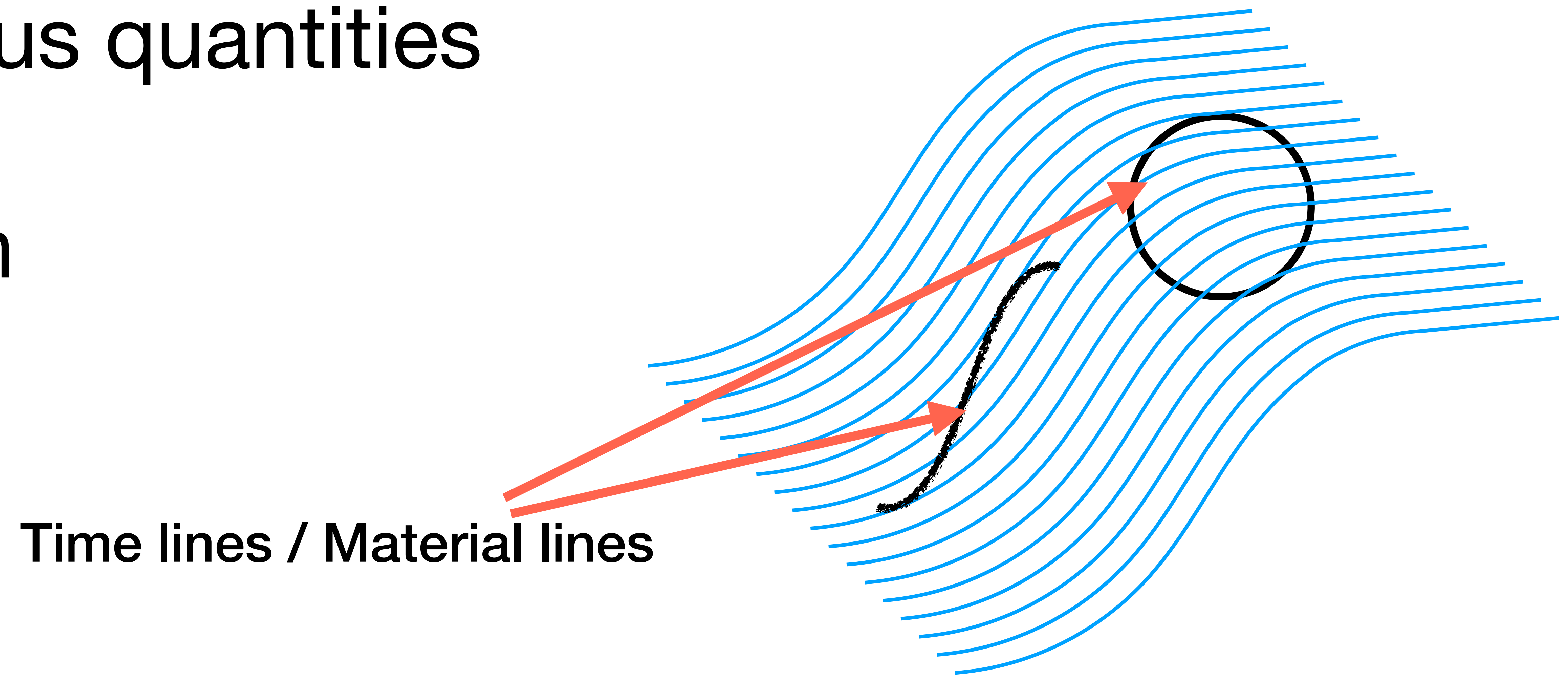
- Conceptual framework of Local Deformation and Rotation Rates
- Material Line and their Kinematics in 2D
- Instantaneous Strain rate and rotation rates of infinitesimal lines

Visualising Local deformation and Rotation Rate: Kinematics of Material Lines

- What is a Material Line / Time Line?

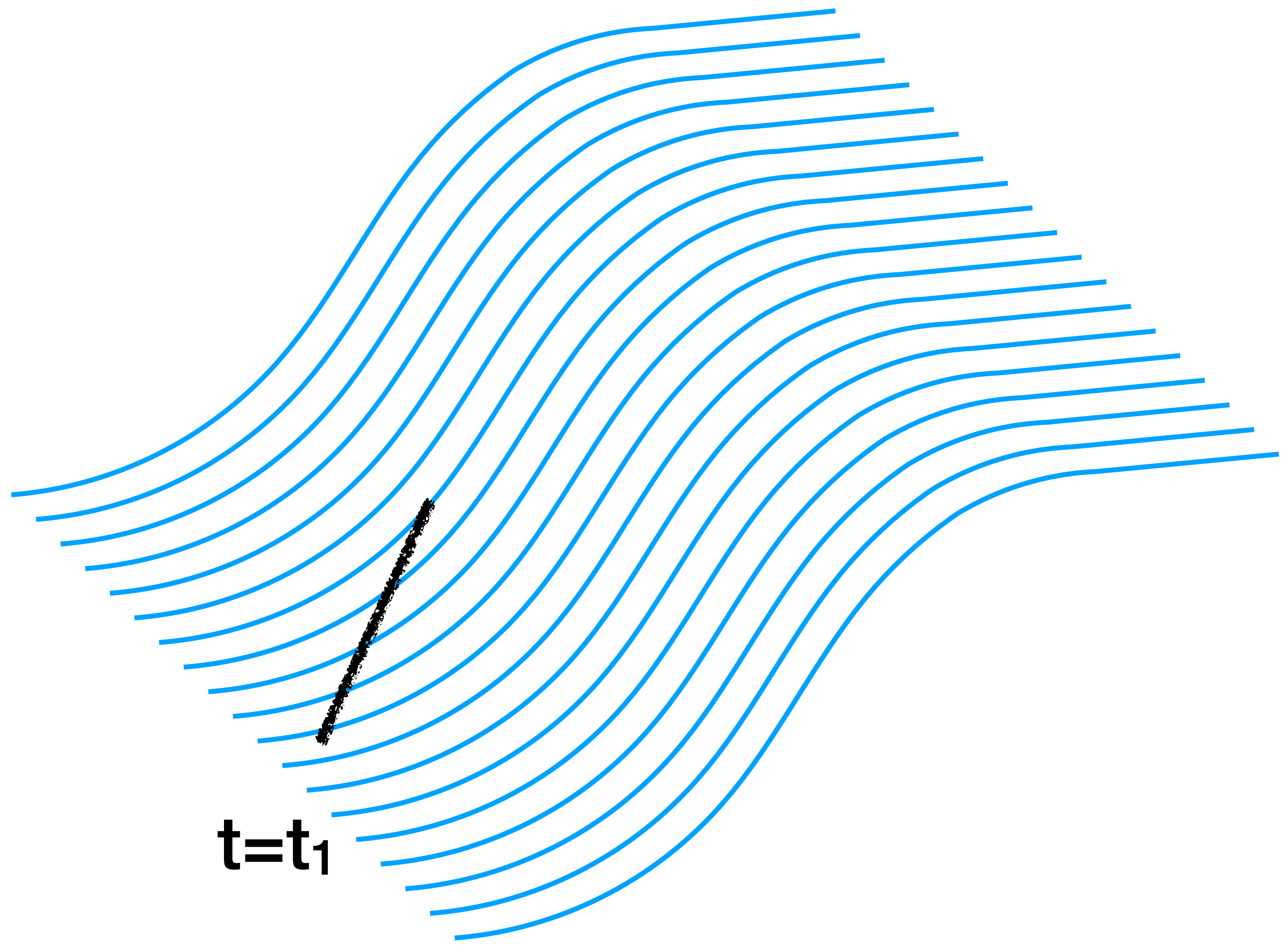
An arbitrary chosen set of fluid particles lying on a curve drawn in the flow domain at a given instant of time.

  - That means they are instantaneous quantities
  - Curve is arbitrary: closed or open



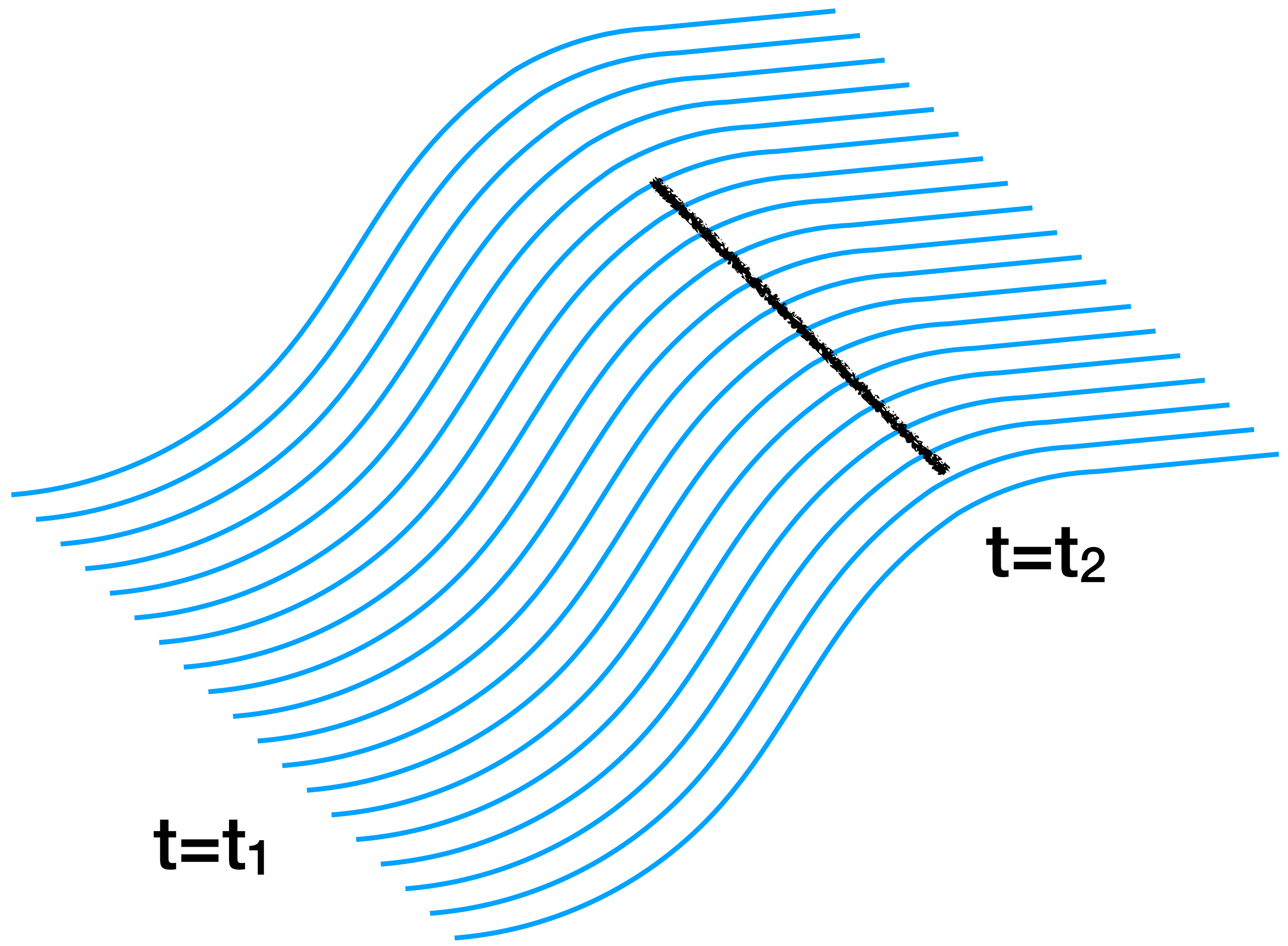
Visualising Local deformation and Rotation Rate

What happens to a material line with the passage of time ?



Visualising Local deformation and Rotation Rate

The above animation shows material line at different instants of time (t_1 and t_2)

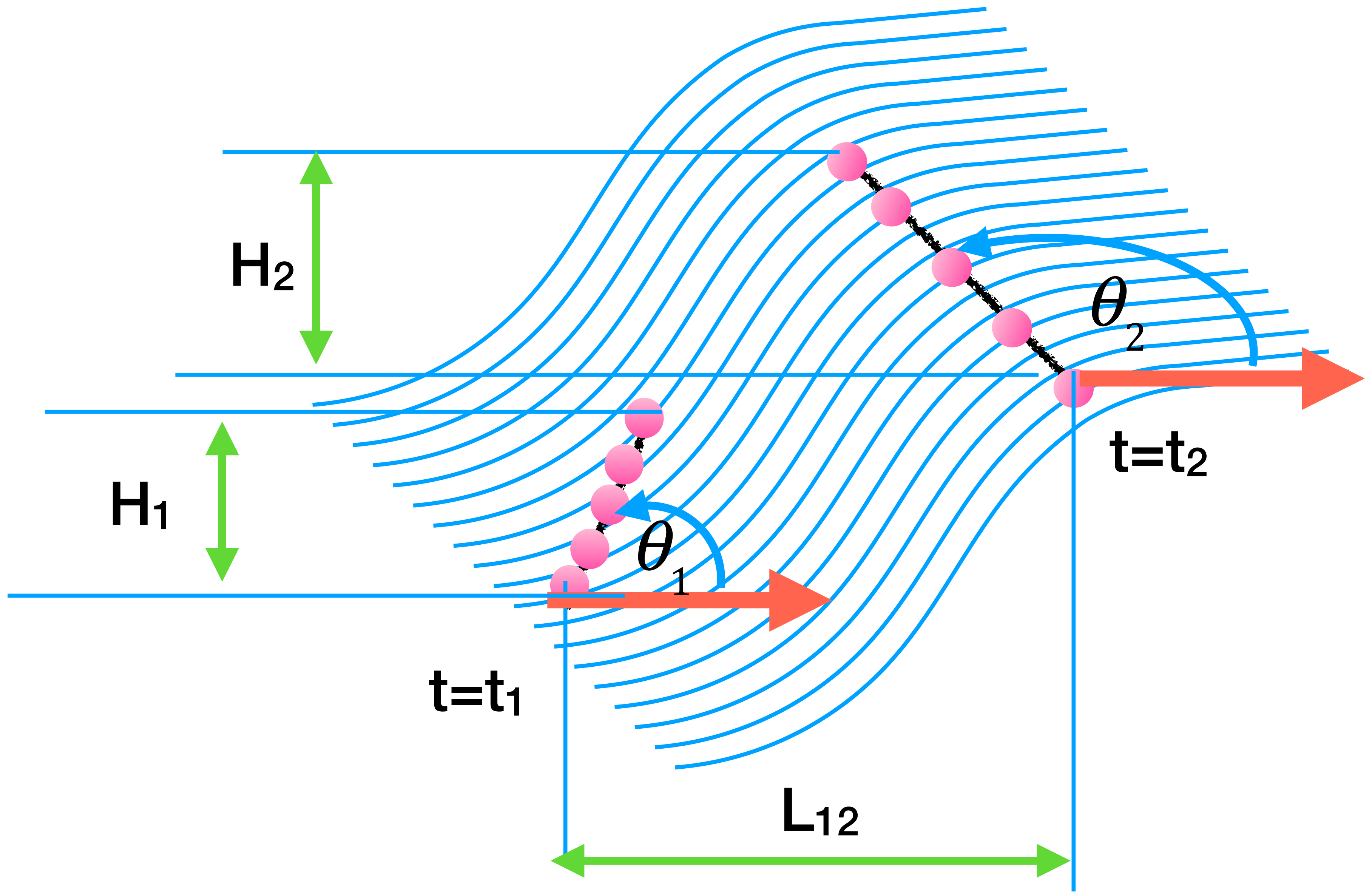


Material line can thus undergo the following kinematic effects:

- Translate
- Rotate
- Longitudinal Stretching (Straining) or Contracting——-Deformation

Visualising Local deformation and Rotation Rate

The above animation shows material line at different instants of time (t_1 and t_2)



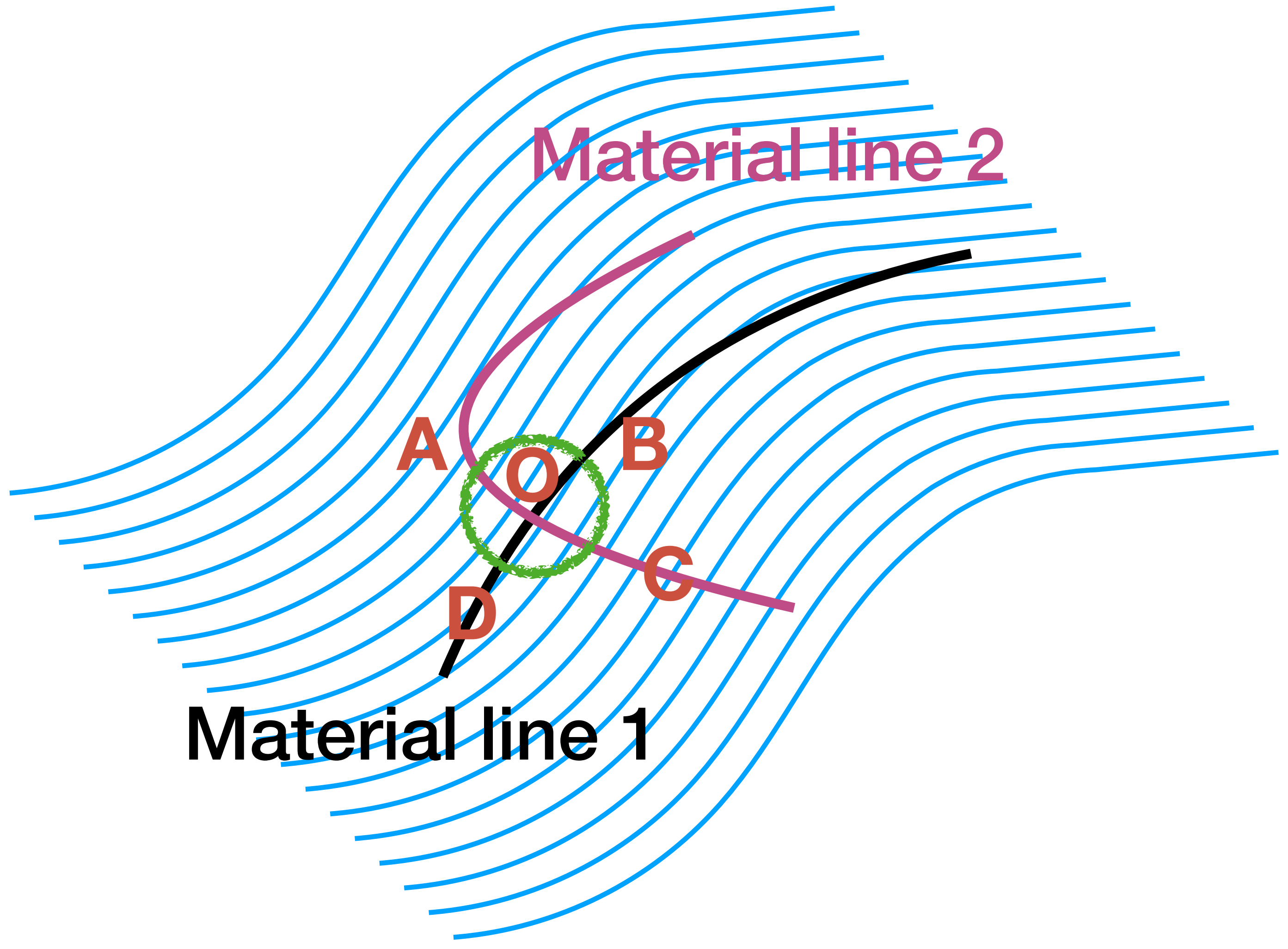
Material line can thus undergo the following kinematic effects:

- Translate (point has moved by L_{12} distance)
- Rotate (Material line has rotated by $\theta_2 - \theta_1 > 0$)
- Longitudinal Stretching (Straining, $H_2 > H_1$) or Contracting— —-Deformation

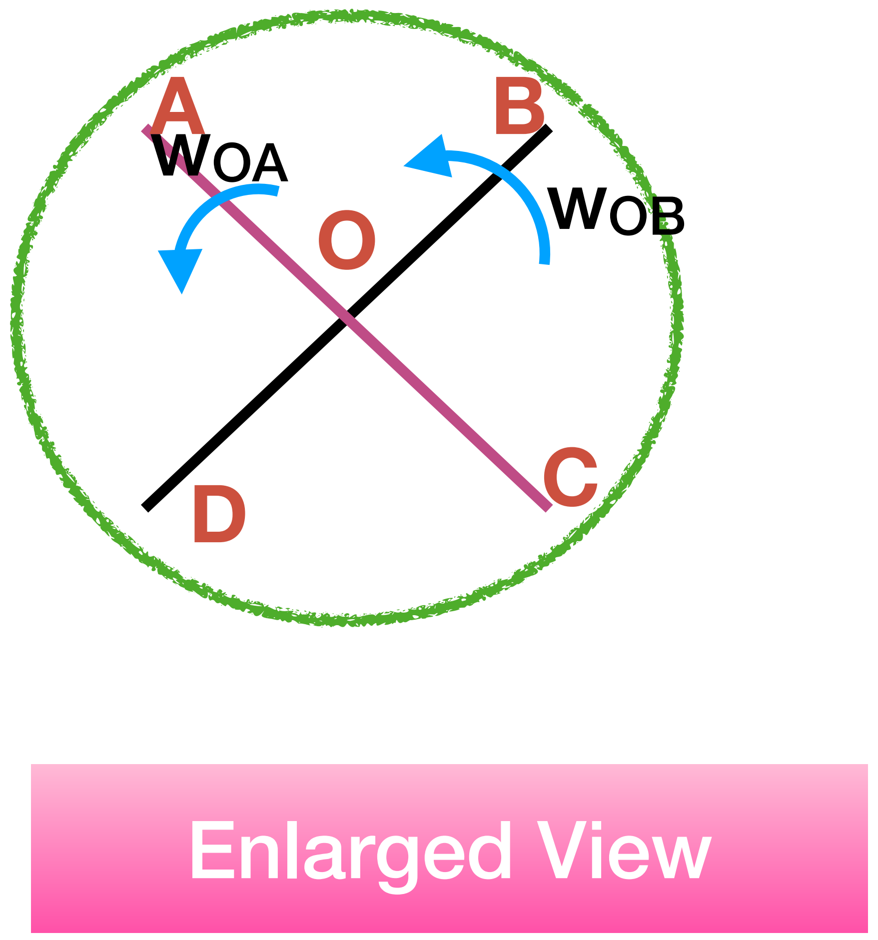
Visualising Shear Rate

- We had earlier visualised Local deformation and rotation rates but not Shear Strain rate

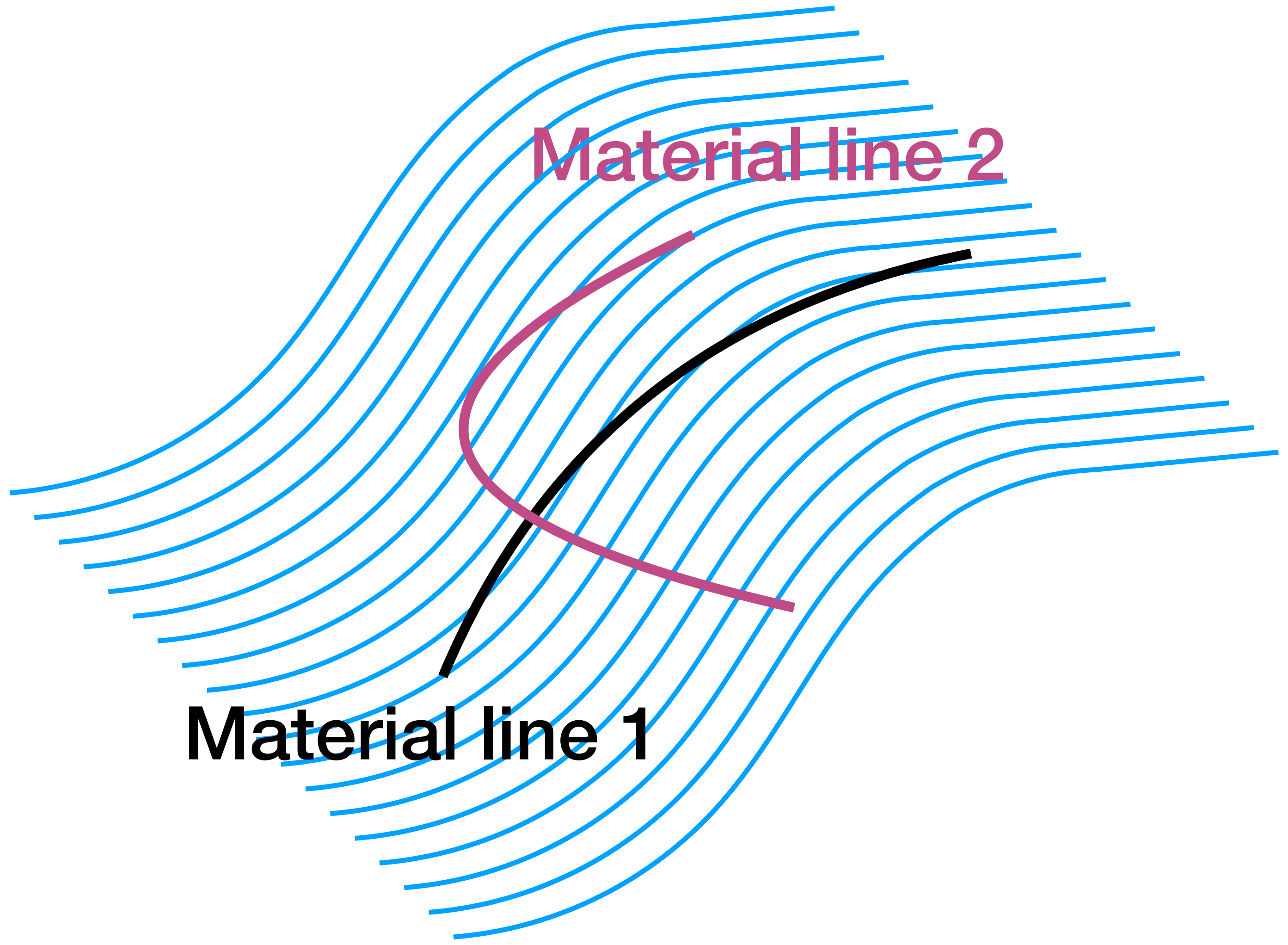
- In order to visualise the shear strain rate, we need to consider two material lines (in fig: Material line 1 and 2), as shear strain is fundamentally defined as rate of angular deformation between a pair of material lines.



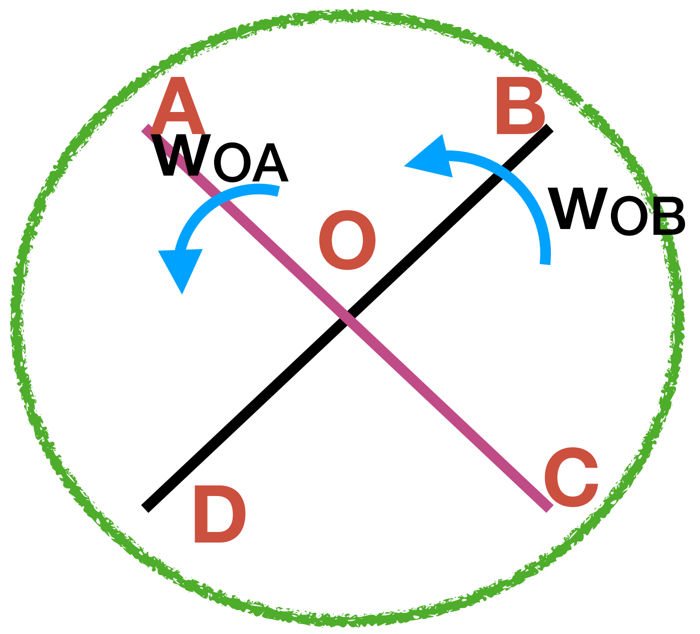
Visualising Shear Rate



- If w_{OA} is different from w_{OB} then angular deformation or shear deformation between OA and OB is taking place.



Visualising Shear Rate



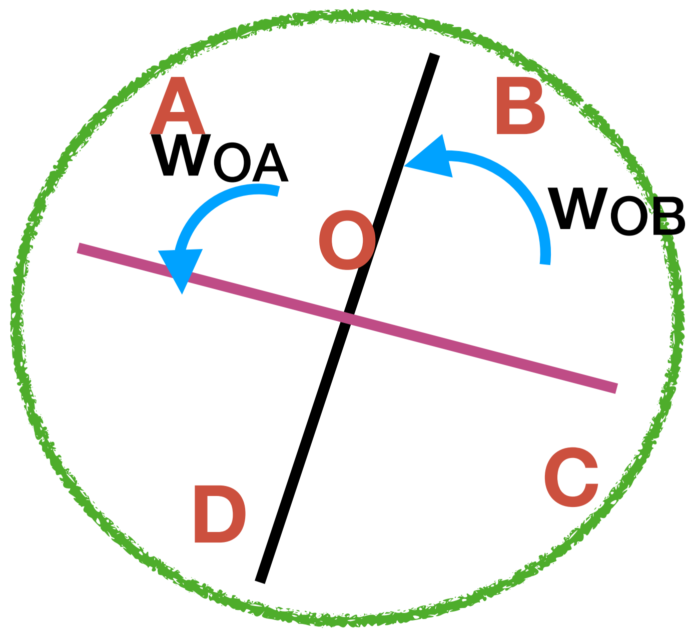
- If w_{OA} is different from w_{OB} then angular deformation or shear deformation between OA and OB is taking place.

Enlarged View

Visualising Shear Rotation Rate

- If w_{OA} is different from w_{OB} then angular deformation or shear deformation between OA and OB is taking place.

- As the angle can only change if w_{OA} is not equal to w_{OB}



Enlarged View

if rotation rates of AC and BD are equal

Visualising Shear Rotation Rate

- If w_{OA} is different from w_{OB} then angular deformation or shear deformation between OA and OB is taking place.
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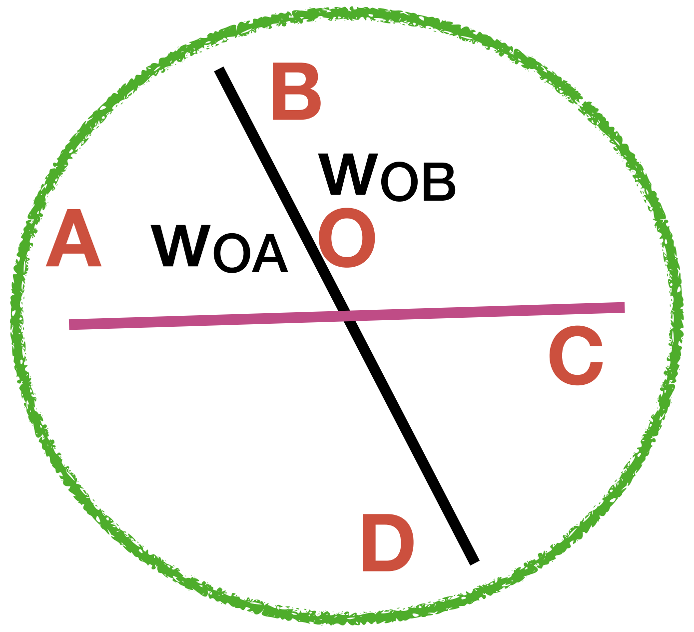


Enlarged View

if rotation rates of AC and BD are not equal

Infinitesimal Local Deformation and Rotation Rate in 2D

$\hat{t} \equiv \text{Cos}\,\theta\hat{i} + \text{Sin}\,\theta\hat{j}$

$\hat{n} \equiv -\text{Sin}\,\theta\hat{i} + \text{Cos}\,\theta\hat{j}$

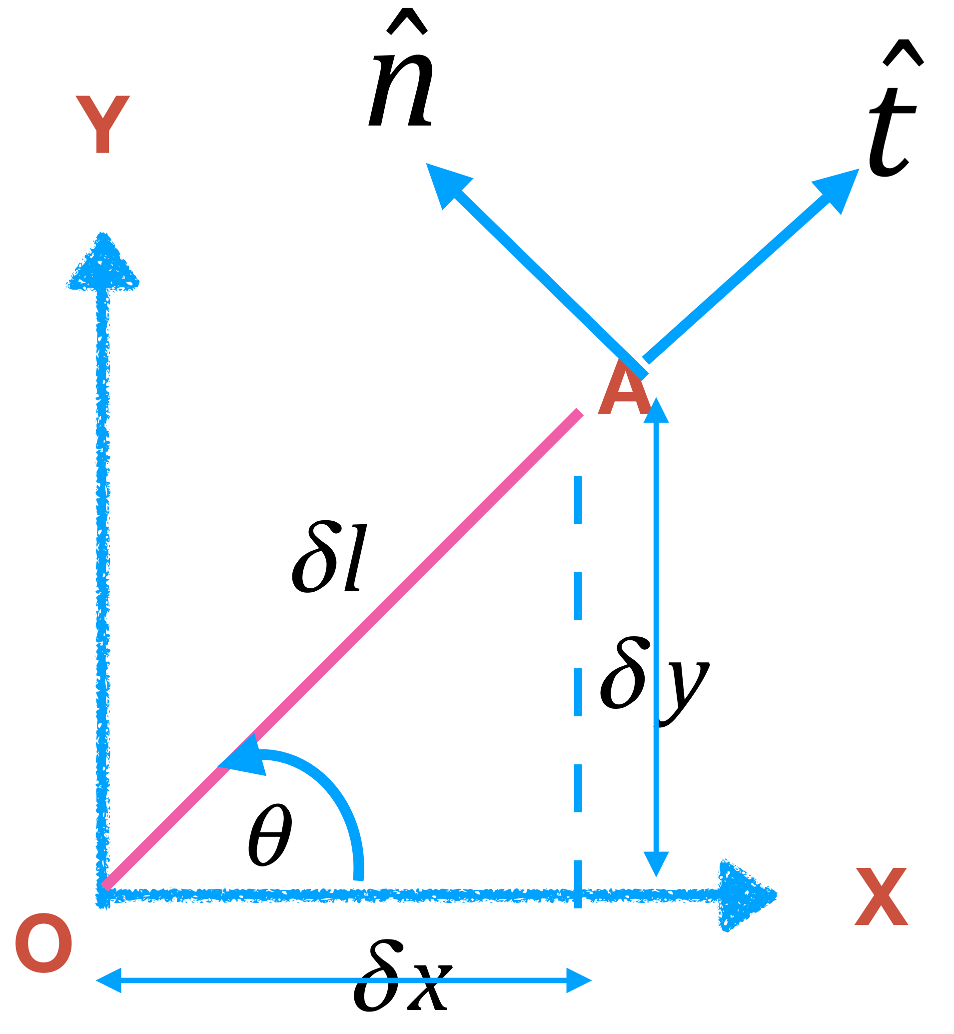
$\vec{V} \equiv u_o\hat{i} + v_o\hat{j}$

In the neighbourhood of point O, as velocity field $u(x,y,t)$ and $v(x,y,t)$

Using Taylor Series expansion, we get

$$u = u_o + \left(\frac{\partial u}{\partial x}\right)_o \delta x + \left(\frac{\partial u}{\partial y}\right)_o \delta y + \ldots\ldots + \text{high order terms}$$

$$v = v_o + \left(\frac{\partial v}{\partial x}\right)_o \delta x + \left(\frac{\partial v}{\partial y}\right)_o \delta y + \ldots\ldots + \text{high order terms}$$

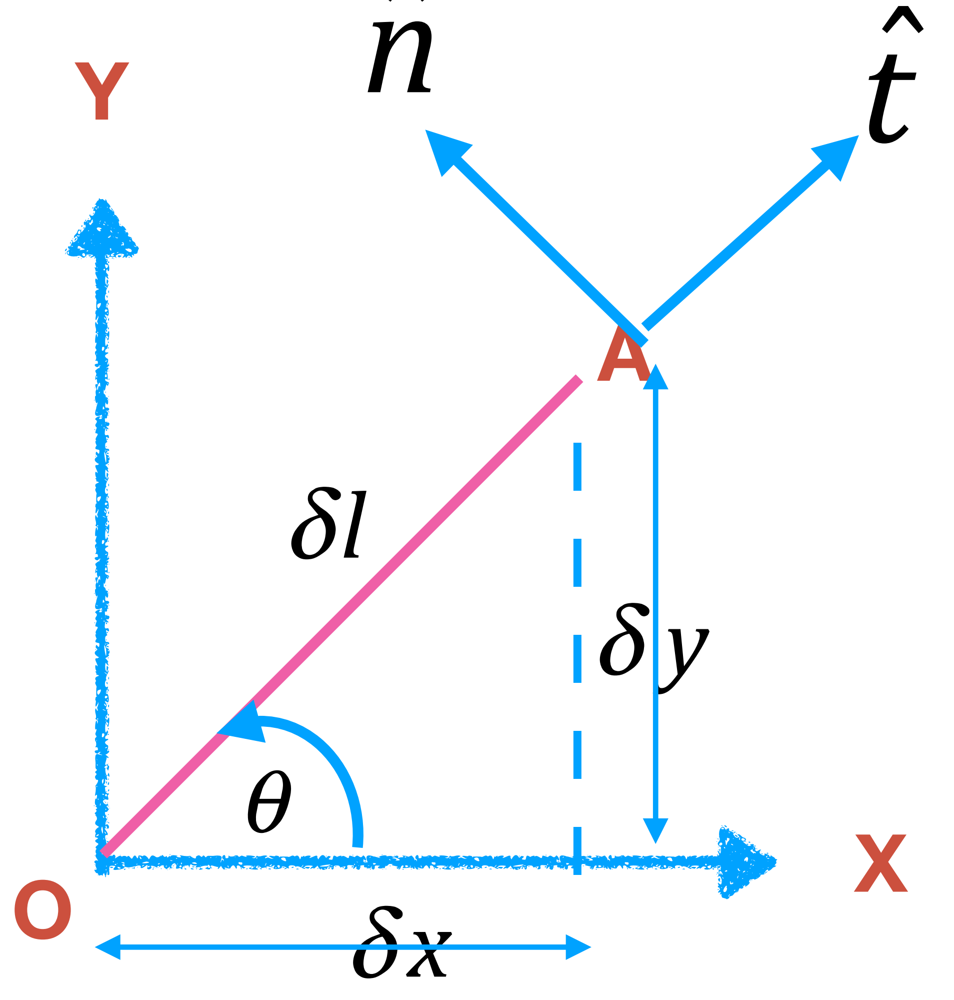


Local Deformation and Rotation Rate in 2D

Longitudnal instantaneous strain rate of OA

(Infinitesimal material line anchored at O, oriented at θ)

and infinitesimal rotation rate of OA can be expressed as



$$\varepsilon_{OA} = \varepsilon_{\theta} = \frac{\vec{V}_{OA}.\hat{t}}{\delta l} = \frac{\left(\vec{V}_A - \vec{V}_O\right).\hat{t}}{\delta l}$$

Responsible for stretching or contraction of OA

$$\omega_{OA} = \omega_{\theta} = \frac{\vec{V}_{OA}.\hat{n}}{\delta l} = \frac{\left(\vec{V}_A - \vec{V}_O\right).\hat{n}}{\delta l}$$

Responsible for rotation of OA

Explanation:

$\left(\vec{V}_A - \vec{V}_O\right).\hat{t} \equiv$ Relative velocity of A with respect to O along OA

$\left(\vec{V}_A - \vec{V}_O\right).\hat{n} \equiv$ Relative velocity of A with respect to O perpendicular to OA

Local Deformation and Rotation Rate in 2D

Completing the simplification using Taylor Series expansion for $\left(u_A - u_O\right)$ and $\left(v_A - v_O\right)$ using δx=δl Cosθ,δy=δl Sinθ

$$\omega_\theta = \left(\frac{\partial v}{\partial x}\right)\mathrm{Cos}^2\theta + \left[\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right)\right]\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta - \left(\frac{\partial u}{\partial y}\right)\mathrm{Sin}^2\theta$$

$$\varepsilon_\theta = \left(\frac{\partial u}{\partial x}\right)\mathrm{Cos}^2\theta + \left[\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right)\right]\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta + \left(\frac{\partial v}{\partial y}\right)\mathrm{Sin}^2\theta$$

These are fundamental expressions for longitudinal instantaneous strain rate and instantaneous rotation strain rate of an infinitesimal material line instantaneously anchored at some point in the flow field at some angle 'θ' w.r.t x- direction

Local Deformation and Rotation Rate in 2D

- Conclusion:

1. Both longitudinal and rotation rate depend upon

   a. the orientation of the infinitesimal material line at some point in the 2D flow domain. $'\theta'$

   b. the local partial derivatives of velocities (u,v) at some point in the flow domain $\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial v}{\partial x}\right), \left(\frac{\partial u}{\partial y}\right) \text{and} \left(\frac{\partial v}{\partial y}\right)$

2. The dependency on the $'\theta'$ in periodic with a period of $'\pi'$,

   i.e. $\omega_{OA} = \omega_{OC}$ and $\varepsilon_{OA} = \varepsilon_{OC}$

Thanks

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Module 1: Basics of Viscous Flows

Lecture 3: Local Deformation and Rotation

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Objectives

- Expression for shear deformation in 2D
- Physical meaning of partial derivatives of velocity
- Properties of Strain rate and rotation rates
- Extension to 3D

Expression for shear deformation in 2D

- Consider a pair of perpendicular infinitesimal material AC and BD intersecting at O. Now Consider these as infinitesimal perpendicular material lines AO and OB anchored at point O, instantaneously oriented as shown in figure.

- The instantaneous rotation rates of OA and OB can be used the shear (angular) deformation rate at point O.

$$\gamma_{OA,OC} = \omega_{OA} - \omega_{OB}$$

$$\omega_{OA(\theta)} = \left(\frac{\partial v}{\partial x}\right)\text{Cos}^2\theta + \left[\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right)\right]\text{Sin}\theta\text{Cos}\theta - \left(\frac{\partial u}{\partial y}\right)\text{Sin}^2\theta$$

Can also be written as

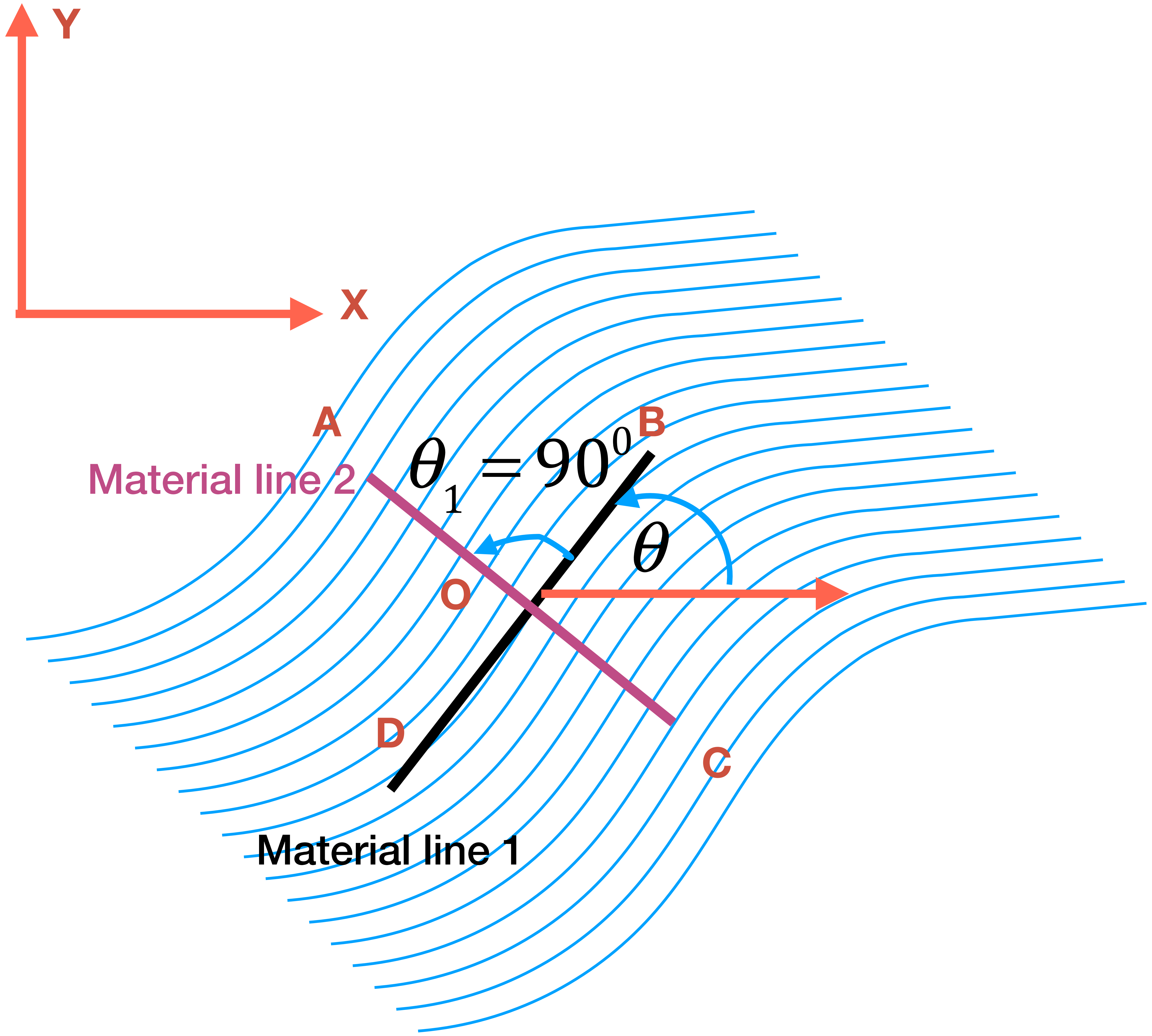
$$\gamma_{OA,OC} = \omega_{OB} - \omega_{OA} \quad \cdots \rightarrow \text{choice of sign convention}$$

we have taken $\gamma_{OA,OC} = \omega_{OA} - \omega_{OB}$ *as* $\gamma > 0$ gives reducing angle

$$\gamma_{\theta,\theta+\frac{\pi}{2}} = \omega_{\theta} - \omega_{\theta+\frac{\pi}{2}}$$

- $$\gamma_{\theta,\theta+\frac{\pi}{2}} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\text{Cos}(2\theta) + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\text{Sin}(2\theta)$$

- The choice of perpendicular infinitesimal material line is arbitrary and is a matter of convention



Physical meaning of partial derivatives of velocity

$$\varepsilon_\theta = \left(\frac{\partial u}{\partial x}\right)\mathrm{Cos}^2\theta + \left[\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right)\right]\mathrm{Sin}\theta\,\mathrm{Cos}\theta + \left(\frac{\partial v}{\partial y}\right)\mathrm{Sin}^2\theta$$

$$\omega_\theta = \left(\frac{\partial v}{\partial x}\right)\mathrm{Cos}^2\theta + \left[\left(\frac{\partial v}{\partial y}\right) - \left(\frac{\partial u}{\partial x}\right)\right]\mathrm{Sin}\theta\,\mathrm{Cos}\theta - \left(\frac{\partial u}{\partial y}\right)\mathrm{Sin}^2\theta$$

$$\gamma_{\theta,\theta+\frac{\pi}{2}} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\mathrm{Cos}(2\theta) + \left(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x}\right)\mathrm{Sin}(2\theta)$$

The expression for $\left[\varepsilon_\theta, \omega_\theta, \gamma_\theta\right]$ allow us to physically interpret the partial derivatives of the velocity

$$\left(\frac{\partial u}{\partial x}\right), \left(\frac{\partial u}{\partial y}\right), \left(\frac{\partial v}{\partial x}\right), \left(\frac{\partial v}{\partial y}\right)$$

Consider,

$$\varepsilon_{\theta=0} = \left(\frac{\partial u}{\partial x}\right) \Leftrightarrow \varepsilon_{xx} = \left(\frac{\partial u}{\partial x}\right), \varepsilon_{\theta=\frac{\pi}{2}} = \left(\frac{\partial v}{\partial y}\right) \Leftrightarrow \varepsilon_{yy} = \left(\frac{\partial v}{\partial y}\right)$$

$$\omega_{\theta=0} = \left(\frac{\partial v}{\partial x}\right) or\ \omega_x \,\&\, \omega_{\theta=\frac{\pi}{2}} = -\left(\frac{\partial u}{\partial y}\right) or\ \omega_y ....\gamma_{0,\frac{\pi}{2}} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \gamma_{x,y}$$

Physical meaning of partial derivatives of velocity

$\left(\frac{\partial u}{\partial x}\right) \equiv$ longitudnal instantaneous strain rate of an infinitesimal material line anchored at the point under consideration aligned along x-direction

$\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right) \equiv$ instantaneous shear strain rate of a pair of mutually perpendicular infinitesimal material lines aligned in x and y direction

$\left(\frac{\partial v}{\partial y}\right) \equiv$ longitudnal instantaneous strain rate of an infinitesimal material line anchored at the point under consideration aligned along y-direction

$\left(\frac{\partial v}{\partial x}\right) \equiv$ instantaneous rotation rate of an infinitesimal material line anchored at the point under consideration aligned along x-direction rotating in counter clockwise direction

$-\left(\frac{\partial u}{\partial y}\right) \equiv$ instantaneous rotation rate of an infinitesimal material line anchored at the point under consideration aligned along y-direction rotating in counter clockwise direction

Properties of local deformation and rotation rates (2D)

- **Maximum / Minimum values:**

  - Since $\varepsilon_\theta, \omega_\theta, \gamma_{\theta,\theta+\frac{\pi}{2}}$ are periodic functions of θ they exhibit maximum and minimum values at a given point and time instant in the flow domain, for example:

$$\varepsilon_\theta = \left(\frac{\partial u}{\partial x}\right)\mathrm{Cos}^2\theta + \left[\left(\frac{\partial u}{\partial y}\right) + \left(\frac{\partial v}{\partial x}\right)\right]\mathrm{Sin}\theta\,\mathrm{Cos}\theta + \left(\frac{\partial v}{\partial y}\right)\mathrm{Sin}^2\theta$$

$$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\mathrm{Sin}2\theta + \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\mathrm{Cos}2\theta$$

The above is an expression of type ($a\sin 2\theta + b\cos 2\theta + c$)

$$\Rightarrow \left(\varepsilon_\theta\right)_{\max} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{2}\sqrt{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2}$$

- $$\Rightarrow \left(\varepsilon_\theta\right)_{\min} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{1}{2}\sqrt{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)^2}$$

These instantaneous maximum and minimum values at a point in the flow domain are called of principal strain rates.

The corresponding principal directions can be found as

$$\frac{d\varepsilon_\theta}{d\theta} = 0 \Rightarrow \tan 2\theta_p = \frac{\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}{\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)}$$

This gives the values for principal directions θ_p and $\theta_{p+\frac{\pi}{2}}$

Properties of local deformation and rotation rates (2D)

- Invariants w.r.t θ

$$\varepsilon_\theta + \varepsilon_{\theta+\frac{\pi}{2}} \equiv \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \equiv \text{Sum of longitudnal strain rates of two mutually perpendicular infinitesimal material lines at a given point and at a certain instant}$$

Does not depend on θ

Physically $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ represents a very important property of fluid particles as shown later ……Hint **Volumetric strain rate**

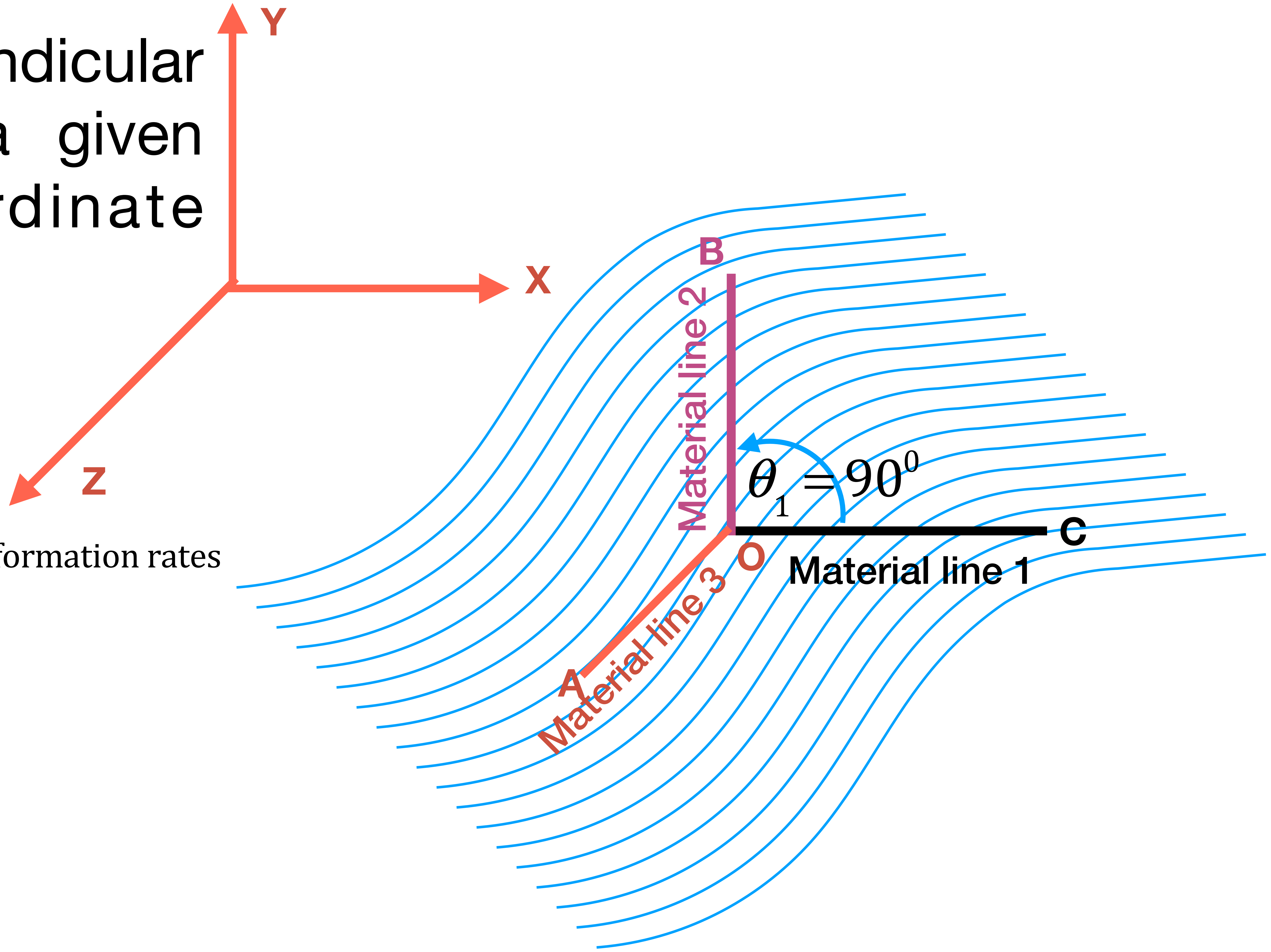
$$\omega_\theta + \omega_{\theta+\frac{\pi}{2}} \equiv \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \equiv \text{Sum of rotation rates of two mutually perpendicular infinitesimal material lines at a given point and at a certain instant}$$

Does not depend on θ

Physically $\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ represents a very important property of fluid particles —-**vorticity**

Extension to 3D

- Consider three mutually perpendicular infinitesimal material lines at a given point along the three coordinate directions:

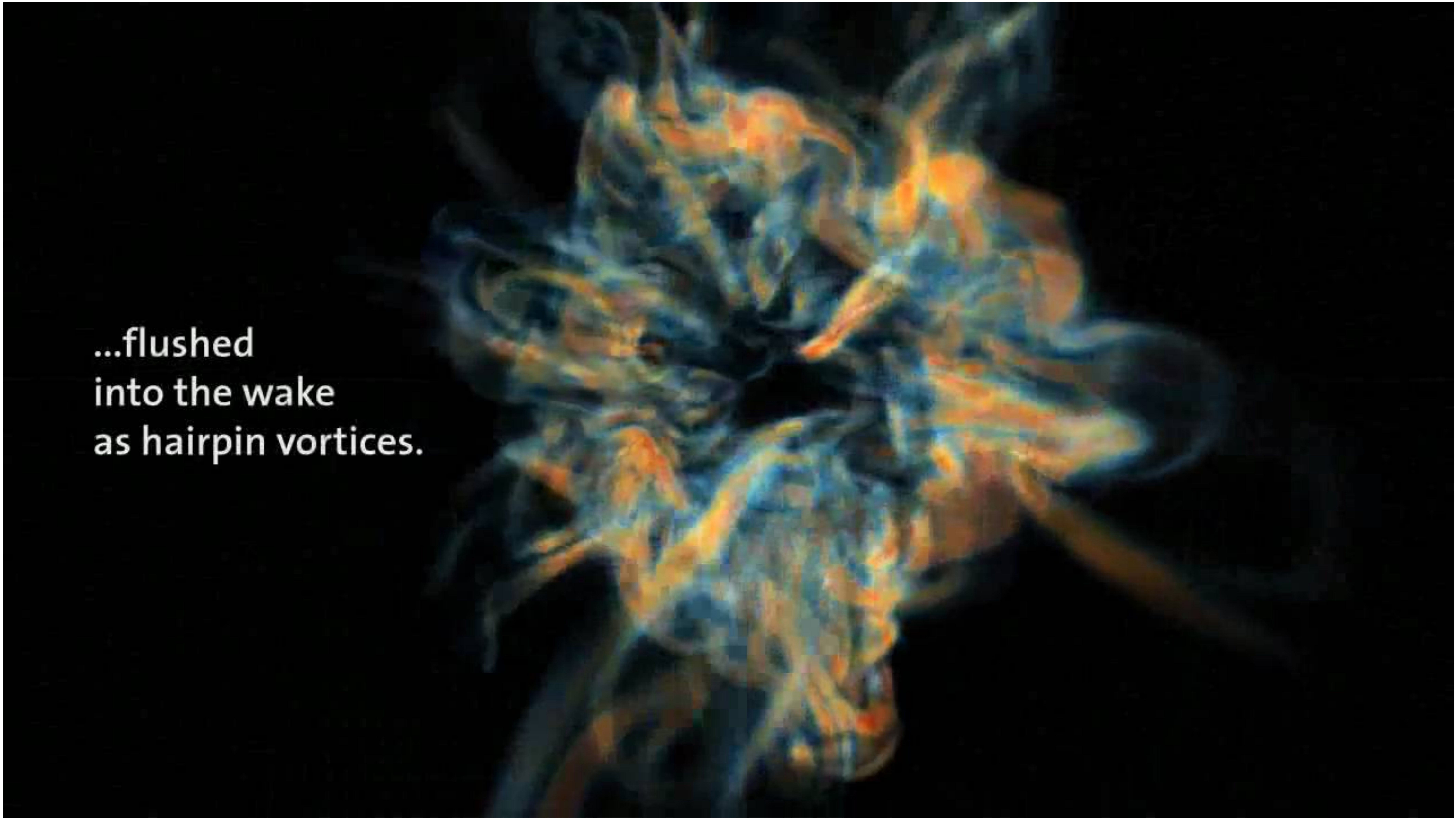


$$\varepsilon_{xx} = \left(\frac{\partial u}{\partial x}\right), \varepsilon_{yy} = \left(\frac{\partial v}{\partial y}\right), \varepsilon_{xx} = \left(\frac{\partial w}{\partial z}\right) \quad \cdots \text{Longitudnal Strain rates}$$

$$\gamma_{xy} = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \gamma_{yz} = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \gamma_{xz} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \quad \cdots \text{Shear deformation rates}$$

- $$\Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \Omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \Omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad \cdots \text{Vorticity}$$

Pair of Perpendicular material lines in x-y plane	Pair of Perpendicular material lines in y-z plane	Pair of Perpendicular material lines in z-x plane



Thanks

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Fluid Mechanics -II (MEC3310)

Module 1: Basics of Viscous Flows

Lecture 4: Local Deformation rates: Volumetric Strain Rate

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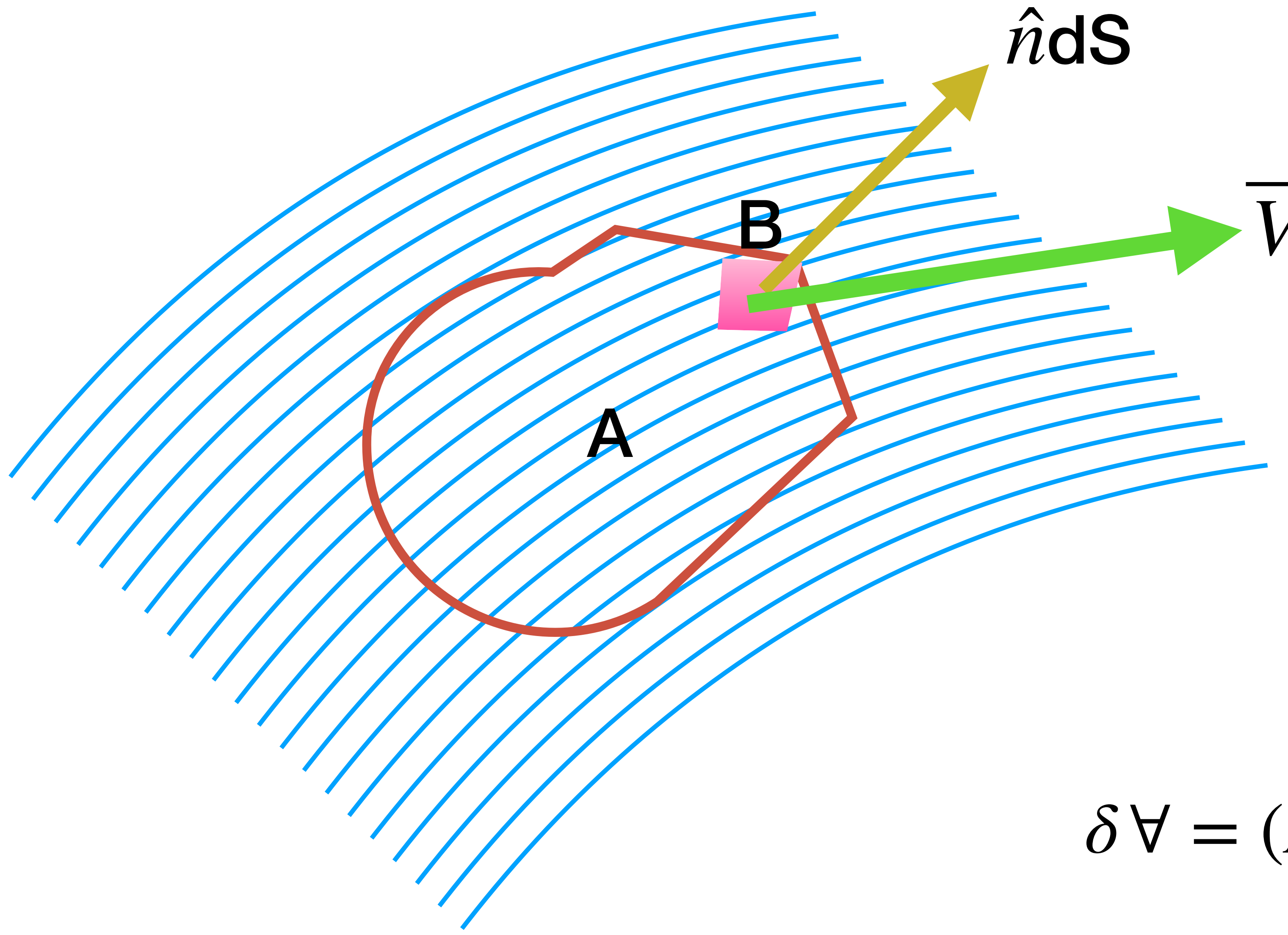
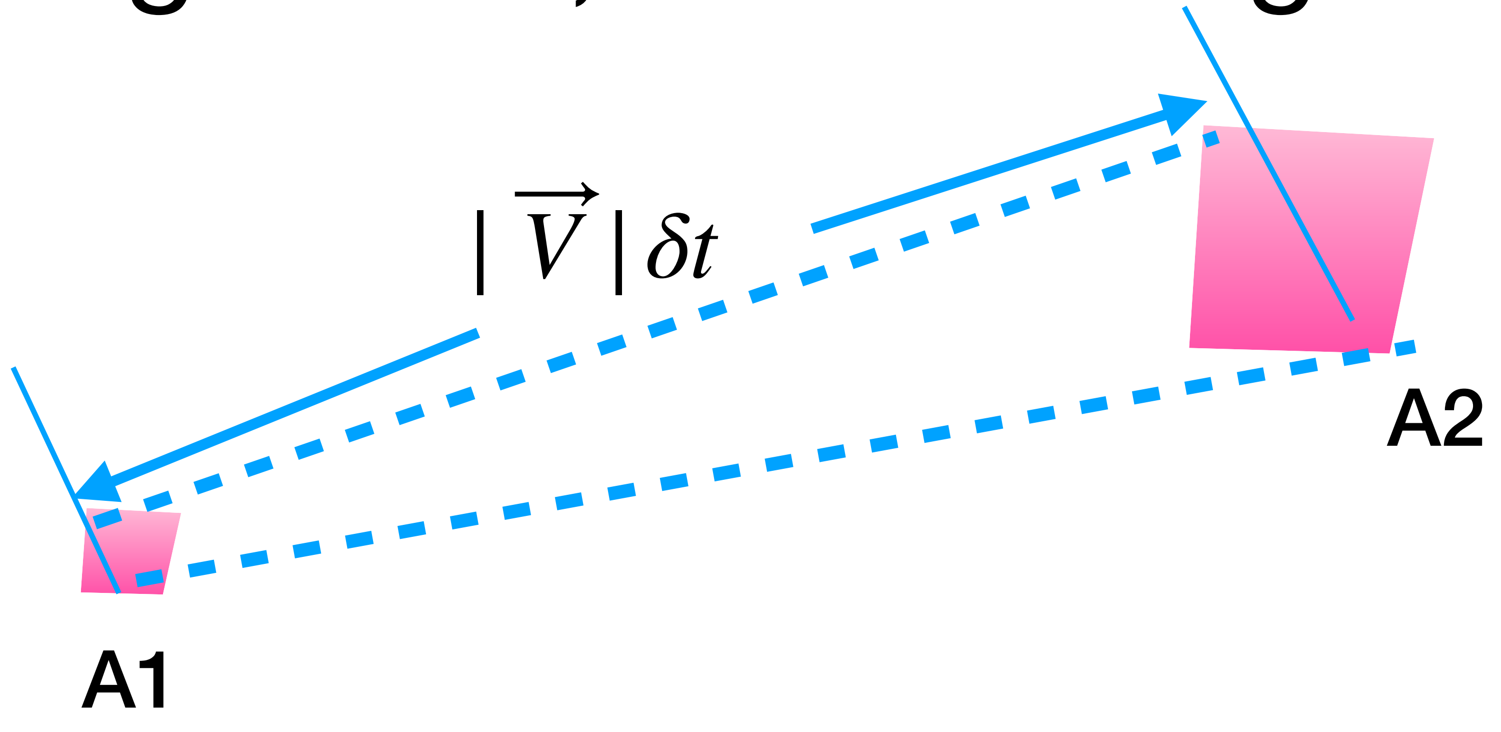
Learning Outcome

- Students will be able to:
  - Obtain the instantaneous volumetric strain rate for a fluid particle in a coordinate free form
  - To express the volumetric strain rate in different coordinate systems- Cartesian and Cylindrical

Volumetric Strain rate

- In order to express the instantaneous volumetric strain rate in a fluid particle at a point, consider a finite region ‘R’, surrounding the point





Volume swept in space in a time interval ‘δt’ at a local point B on the surface of ‘R’

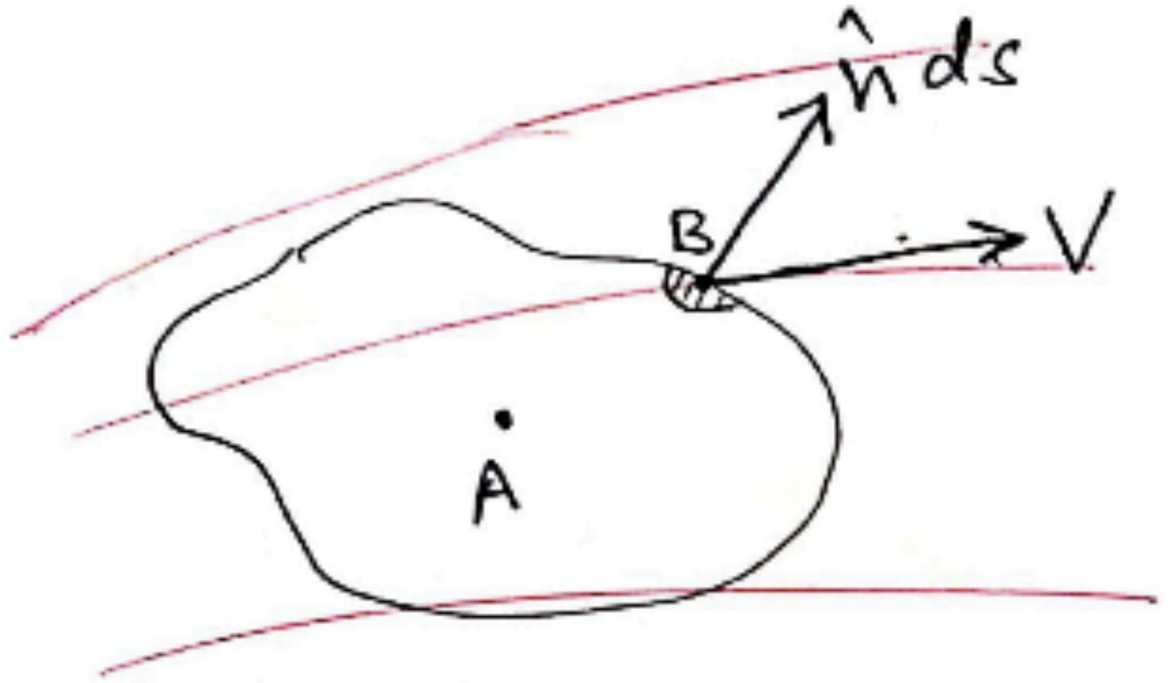
$$\delta \forall = (MeanCrossSectionalArea) . \vec{V}\delta t = \frac{\vec{A}_1 + \vec{A}_2}{2} . (\delta t \langle \vec{V} \rangle)$$

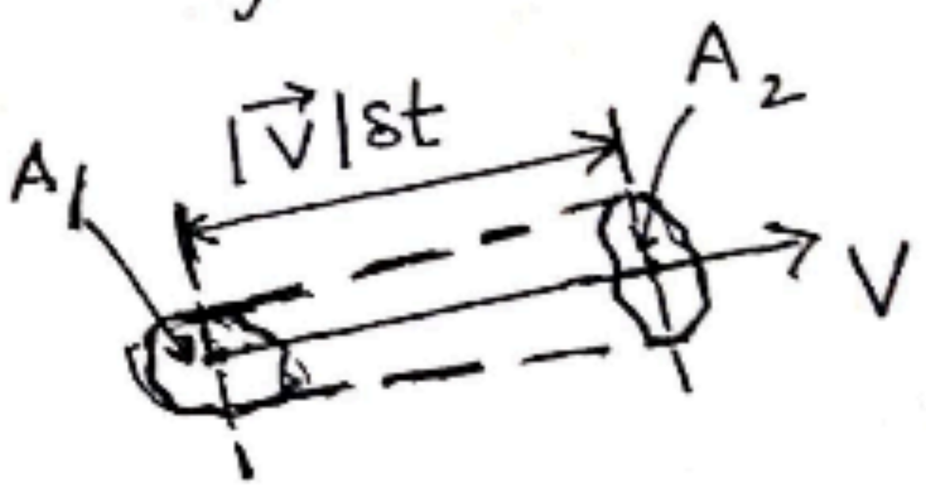
$$\vec{A}_2 = \vec{A}_1 + \delta \vec{A}_1 = (\vec{A}_1 + \frac{\delta \vec{A}_1}{2}) . \langle \vec{V} \rangle$$

Volumetric Strain rate

Volumetric Strain rate

In order to express the instantaneous volumetric strain rate in a fluid particle at a point, consider a finite ~~region~~ region 'R' surrounding the point.





Volume swept in space in a time interval 'δt' at a local pt 'B' on the surface of 'R'.

$(\delta \forall)_{\substack{elem.\\ at\ B}}$ = (Mean X-sectional Area) $\cdot$ $(\vec{V}\,\delta t)$ dot prod.

$$= \left[\frac{\vec{A}_1 + \vec{A}_2}{2}\right]\cdot\langle\vec{V}\rangle\delta t$$

Now $\vec{A}_2 = \vec{A}_1 + \delta\vec{A}$

$$= \left[\vec{A}_1 + \frac{\delta\vec{A}}{2}\right]\cdot\langle\vec{V}\rangle\delta t$$

$$= \vec{A}_1\cdot\langle\vec{V}\rangle\delta t + \frac{\delta\vec{A}}{2}\cdot\langle\vec{V}\rangle\delta t$$

$$= ds\,\hat{n}\cdot\langle\vec{V}\rangle\delta t + \frac{\delta\vec{A}}{2}\cdot\langle\vec{V}\rangle\delta t$$

Volumetric Strain rate

Thus, the change in volume of region 'R' in a small time interval δt is given as,

$$(\Delta \forall)_R = \oint\!\!\oint_S \langle \vec{V} \rangle \cdot \hat{n}\, ds\, \delta t + \oint\!\!\oint \frac{\vec{\delta A} \cdot \langle \vec{V} \rangle \delta t}{2} ds$$

Volumetric strain rate of region R

$$= \lim_{\delta t \to 0} \frac{(\Delta \forall)_R}{\forall\, \delta t}$$

$$= \lim_{\delta t \to 0} \frac{1}{\forall} \left[\oint\!\!\oint_S \langle \vec{V} \rangle \cdot \hat{n}\, ds + \oint\!\!\oint \frac{\vec{\delta A} \cdot \langle \vec{V} \rangle}{2} ds \right]$$

In the limit as $\delta t \to 0$, $\langle \vec{V} \rangle \to \vec{V}$

" " " " $\delta t \to 0$, $\vec{\delta A} \to 0$

$$(\epsilon_v)_R = \frac{1}{\forall} \oint\!\!\oint_S \vec{V} \cdot \hat{n}\, ds \longrightarrow \text{Instantaneous volumetric strain rate of region 'R'}$$

The surface integral may be replaced by volume integral using Gauss theorem from vector calculus.

$$\underbrace{\iiint_R (\nabla \cdot \vec{V})\, d\forall}_{\text{divergence of } \vec{V}} = \oint\!\!\oint_S \vec{V} \cdot \hat{n}\, ds$$

$$(\epsilon_v)_R = \frac{1}{\forall} \iiint_R (\nabla \cdot \vec{V})\, d\forall$$

In order to estimate the instantaneous volumetric strain rate of a fluid particle at A, we take the limit of $(\epsilon_v)_R$ as $\forall \to d\forall$.

$$(\epsilon_v)_A = \lim_{\forall \to d\forall} \frac{1}{\forall} \iiint_R (\nabla \cdot \vec{V})\, d\forall$$

$$= \frac{(\nabla \cdot \vec{V})_A\, d\forall}{d\forall} = (\nabla \cdot \vec{V})_A$$

Thus, in general volumetric strain rate at any pt in the flow domain is $\underbrace{(\nabla \cdot \vec{V})}_{\text{divergence of } \vec{V}}$

$$\nabla \equiv \text{del operator} \equiv \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$$

Volumetric Strain rate

Procedure for applying the ∇-operator.

In cartesian coordinate system

$$\nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(u\hat{i}+v\hat{j}+w\hat{k}\right)$$

$$= \hat{i}\cdot\left[\frac{\partial(u\hat{i}+v\hat{j}+w\hat{k})}{\partial x}\right] + \hat{j}\cdot\left[\frac{\partial(u\hat{i}+v\hat{j}+w\hat{k})}{\partial y}\right] + \hat{k}\cdot\left[\frac{\partial(u\hat{i}+v\hat{j}+w\hat{k})}{\partial z}\right]$$

$$= \hat{i}\cdot\left[\frac{\partial u}{\partial x}\hat{i} + \frac{\partial v}{\partial x}\hat{j} + \frac{\partial w}{\partial x}\hat{k}\right] + \hat{j}\cdot\left[\frac{\partial u}{\partial y}\hat{i} + \frac{\partial v}{\partial y}\hat{j} + \frac{\partial w}{\partial y}\hat{k}\right] + \hat{k}\cdot\left[\frac{\partial u}{\partial z}\hat{i} + \frac{\partial v}{\partial z}\hat{j} + \frac{\partial w}{\partial z}\hat{k}\right]$$

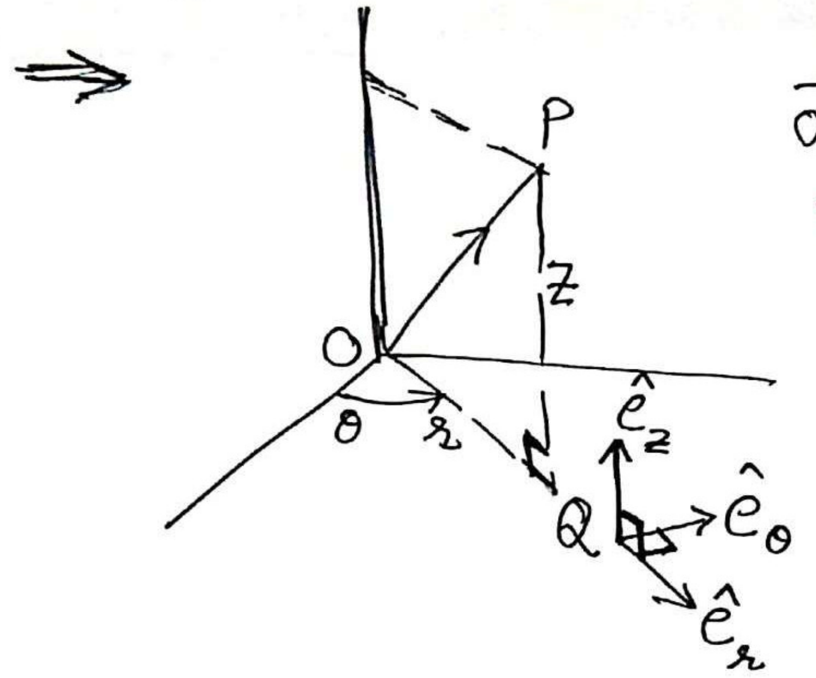
$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$= \left(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}\right)$$

= sum of the three longitudinal strain rates

In cylindrical coordinate system

$\Rightarrow$



$$\vec{OP} = \vec{r} = r\hat{e}_r + z\hat{e}_z$$

There is no θ-component in the position vector

In cylindrical coordinates system, unit vectors $\hat{e}_r$, $\hat{e}_\theta$ change with θ.

It can be shown that

$$\frac{\partial\hat{e}_r}{\partial\theta} = \frac{d\hat{e}_r}{d\theta} = \hat{e}_\theta$$

$$\frac{\partial\hat{e}_\theta}{\partial\theta} = \frac{d\hat{e}_\theta}{d\theta} = -\hat{e}_r$$

$$\nabla\cdot\vec{V} = \left(\hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}\right)\cdot\left(V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{e}_z\right)$$

$$= \underbrace{\left(\frac{\partial V_r}{\partial r}\right) + \left(\frac{V_r}{r}\right)} + \left(\frac{1}{r}\frac{\partial V_\theta}{\partial\theta}\right) + \left(\frac{\partial V_z}{\partial z}\right) \rightarrow$$

Try to do yourself

$$= \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\partial V_z}{\partial z}.$$

Algebra of Scalar, Vector and Tensors

Learning objectives:

- Representation of vectors, tensors in general coordinate systems using Indical notation
- Handling algebraic operations in general and orthogonal coordinate systems

Algebra & Calculus of Scaler, Vector and Tensor functions

In fluid Mechanics we deal with

1. Scalers : density, temp, Kinetic energy, pressure etc.
2. Vectors : Velocity, acceleration, forces, Vorticity
3. Tensors : strain rates at a point, stress at a point

Scalars → Magnitude only

Vectors → Magnitude + direction

Tensors of rank 2 → Magnitude + $\boxed{\text{direction\_1} + \text{direction\_2}}$

Notations

scaler field = $\phi(\vec{r}, t)$

Vector field = $\vec{A}(\vec{r}, t)$

Tensor field = $\overline{\overline{T}}(\vec{r}, t)$

Unit vectors and Vector, Tensor representation

$$\vec{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$

$\hat{e}_1, \hat{e}_2, \hat{e}_3$ → unit vectors along coordinate directions

We are working with a general coordinate system in mind.

Indical notation

$$\vec{A} = \sum_1^3 A_i\hat{e}_i$$

$\vec{A} = A_i\hat{e}_i$ (Summation is implied over i → a repeating index)

i is repeating in the expression

$\overline{\overline{T}} = T_{ij}\hat{e}_i\hat{e}_j$ (Summation over i and j is implied)

Each component has a mag. + 2 dirs.

$$\begin{aligned} = T_{11}\hat{e}_1\hat{e}_1 + T_{12}\hat{e}_1\hat{e}_2 + T_{13}\hat{e}_1\hat{e}_3 &+ \\ T_{21}\hat{e}_2\hat{e}_1 + T_{22}\hat{e}_2\hat{e}_2 + T_{23}\hat{e}_2\hat{e}_3 &+ \\ T_{31}\hat{e}_3\hat{e}_1 + T_{32}\hat{e}_3\hat{e}_2 + T_{33}\hat{e}_3\hat{e}_3 & \end{aligned} \Bigg\} \text{ Nine components in 3D}$$

Unit dyads

Algebra Operations

→ Addition / Subtraction

- Two vectors ⇒ $\vec{A} + \vec{B} = (A_i + B_i)\hat{e}_i$
- Two tensors ⇒ $\overline{\overline{T}}_1 + \overline{\overline{T}}_2 = [(T_{ij})_1 + (T_{ij})_2]\hat{e}_i\hat{e}_j$

Add / subtract the corresponding components.

→ The inner product (Dot product)

- Two vectors

$\vec{A}\cdot\vec{B} \Rightarrow (A_i\hat{e}_i)\cdot(B_j\hat{e}_j)$

or

$$\boxed{\vec{A}\cdot\vec{B} = A_i B_j (\hat{e}_i\cdot\hat{e}_j)}$$ [Summation over i and j]

→ For a general coordinate system

Result is a scalar

We often use, orthogonal coordinate systems.

$$\Rightarrow \hat{e}_i\cdot\hat{e}_j = 1 \quad i=j$$
$$= 0 \quad i \neq j$$

examples → Cartesian, Cylindrical

For orthogonal system:

$$\hat{e}_i\cdot\hat{e}_j = \delta_{ij} \rightarrow \text{Kronecker delta.}$$

$$\delta_{ij} = 1 \quad i=j$$
$$= 0 \quad i\neq j$$

$$\vec{A}\cdot\vec{B} = A_i B_j \delta_{ij}$$ (Summation over i and j)

$$= A_i B_i$$ [∵ $i\neq j$ does not contribute]

Summation over i

$$\boxed{\vec{A}\cdot\vec{B} = A_i B_i}$$

→ For orthogonal system

Multiply the corresponding components and add.

- A Vector and a Tensor (rank 2)

~~$\vec{A}\cdot\overline{\overline{T}} = (A_i\hat{e}_i)\cdot(T_{jk}$~~

$$\vec{A}\cdot\overline{\overline{T}} = (A_i\hat{e}_i)\cdot(T_{jk}\hat{e}_j\hat{e}_k)$$

$$= A_i T_{jk}(\hat{e}_i\cdot\hat{e}_j)\hat{e}_k$$

(Summation over i, j and k is implied)

The result is a vector as it involves only $\hat{e}_k$

For vectors $\longrightarrow$ $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (Commutative)

For vector & Tensor

$\longrightarrow$ $\vec{A}\cdot\overline{\overline{T}} \neq \overline{\overline{T}}\cdot\vec{A}$

$\longrightarrow$ $\vec{A}\cdot\overline{\overline{T}} = \overline{\overline{T}}\cdot\vec{A}$ only if $\overline{\overline{T}}$ is symmetric i.e. $T_{ij} = T_{ji}$

Do it yourself

$\longrightarrow$ Outer product (Cross-product)

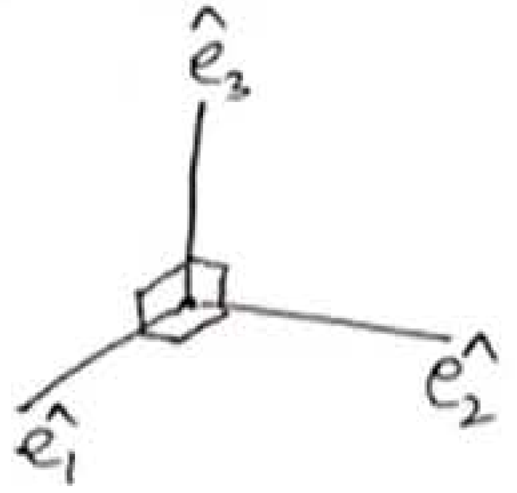
• Two vectors

$$\vec{A}\times\vec{B} = (A_i\,\hat{e}_i)\times(B_j\,\hat{e}_j)$$

$$= A_i B_j\,(\hat{e}_i\times\hat{e}_j)$$

Summation over i and j

For an orthogonal right-hand system of unit vectors as shown below, we have $\hat{e}_i\times\hat{e}_j = \hat{e}_k$ for cyclic $[i, j, k]$ — (1,2,3), (2,3,1), (3,1,2)



Meaning $\Longrightarrow$

$$\hat{e}_1\times\hat{e}_2 = \hat{e}_3$$
$$\hat{e}_2\times\hat{e}_3 = \hat{e}_1$$
$$\hat{e}_3\times\hat{e}_1 = \hat{e}_2$$

By right hand rule

We also have,

$$\hat{e}_i\times\hat{e}_j = -\hat{e}_k \quad \text{for anti-cyclic } [i,j,k]$$

(3,2,1), (1,3,2), (2,1,3)

Finally using a compact notation

$$\boxed{\vec{A}\times\vec{B} = A_i B_j\, e_{ijk}\,\hat{e}_k}$$

Summation over i, j and k

the symbol e_{ijk} is known as permutation symbol defined as

$e_{ijk} = +1$ for (i, j, k) being cyclic

$= -1$ for anti-cyclic (i, j, k)

$= 0$

Since $e_{ijk} = 0$ unless $i \neq j \neq k$

$\Rightarrow$ we have only six terms in the expansion $A_i B_j\, e_{ijk}\,\hat{e}_k$. $(i,j,k) \equiv (1,2,3), (2,3,1), (3,1,2), (1,3,2), (2,1,3), (3,2,1)$

We can define cross-product of a vector with a Tensor of rank 2 $\vec{A}\times\overline{\overline{T}}$ in a similar way as the dot product $\vec{A}\cdot\overline{\overline{T}}$.

$\vec{A}\times\overline{\overline{T}}$ $\longrightarrow$ is a Tensor of rank 2

$\vec{A}\cdot\overline{\overline{T}}$ $\longrightarrow$ is a vector

Other Types of products

1. Dyadic Product of two vectors.

$$\vec{A}\vec{B} = (A_i \hat{e}_i)(B_j \hat{e}_j)$$

$$= A_i B_j \underbrace{\hat{e}_i \hat{e}_j}_{\text{unit dyads}} \rightarrow \text{Tensor of rank 2}$$

In fluid Mechanics, an example of dyadic product is the ~~velocity~~ ~~correlation tensor~~ momentum flux tensor at a point, given as,

$$\rho \vec{V}\vec{V} = \rho(u_i \hat{e}_i)(u_j \hat{e}_j)$$

$$= \rho u_i u_j \hat{e}_i \hat{e}_j$$

2. Tensor dot products

$$\overline{\overline{P}} = P_{ij} \hat{e}_i \hat{e}_j, \quad \overline{\overline{Q}} = Q_{rs} \hat{e}_r \hat{e}_s$$

$$\overline{\overline{P}} \cdot \overline{\overline{Q}} = P_{ij} \hat{e}_i (\hat{e}_j \cdot \hat{e}_r) Q_{rs} \hat{e}_s$$

$$= P_{ij} Q_{rs} (\hat{e}_j \cdot \hat{e}_r) \hat{e}_i \hat{e}_s$$

(summation over i, j, r, s).

Result is tensor of rank 2

For orthogonal systems,

$$\cancel{\overline{\overline{P}} \cdot \overline{\overline{Q}}} \quad \overline{\overline{P}} \cdot \overline{\overline{Q}} = P_{ij} Q_{rs} \delta_{jr} \hat{e}_i \hat{e}_s$$

$$= P_{ij} Q_{js} \hat{e}_i \hat{e}_s$$

(Summation over i, j, s)

A double dot product is also defined for two tensors of rank 2 as,

$$\overline{\overline{P}} : \overline{\overline{Q}} = \underbrace{P_{ij} Q_{rs} (\hat{e}_j \cdot \hat{e}_r)(\hat{e}_i \cdot \hat{e}_s)}_{\text{scalar}}$$

For orthogonal systems,

$$= P_{ij} Q_{rs} \delta_{jr} \delta_{is}$$

$$= P_{ij} Q_{ji} \quad \text{(Summation over } i \text{ and } j\text{)}$$

x ———— x

Algebra and Calculus of Scalar, Vector and Tensor Functions

By

Dr. Nadeem Hasan

1. Scalars, Vectors and Tensors

A scalar is a quantity characterized by a magnitude only e.g. density, temperature, mass, pressure, volume etc. in fluids, any quantity/property in general is a function of location and time. Thus a scalar property in a fluid would behave as a scalar function.

$$S = S(\vec{r}, t) \qquad \text{M.1}$$

A vector is a quantity characterized by a magnitude and direction e.g. velocity, displacement, acceleration, force etc. In fluids a vector property would behave as a

$$\vec{V} = \vec{V}(\vec{r}, t) \qquad \text{M.2}$$

A vector can always be expressed in terms of its components along the coordinate directions.

$$\vec{A} = \sum A_i \hat{e}_i \quad \text{or simply}$$

$$\vec{A} = A_i \hat{e}_i \qquad (\textbf{summation over repeated index i is implied}) \qquad \text{M.3}$$

Where $\hat{e}_i$: unit vector along the i^{th} coordinate direction.

A tensor of second rank is a quantity characterized by a magnitude and two directions e.g. stress, strain rate, mass moment of inertia of a rigid body etc.

Again for fluids, the tensor property would behave as

$$\overline{\overline{T}} = \overline{\overline{T}}(\vec{r}, t) \qquad \text{M.4}$$

In terms of components along the coordinate directions at any point

$$\overline{\overline{T}} = \sum_i \sum_j T_{ij} \hat{e}_i \hat{e}_j \quad \text{or simply}$$

$$\overline{\overline{T}} = T_{ij} \hat{e}_i \hat{e}_j \qquad \text{M.5}$$

The unit vector pairs $\hat{e}_i \hat{e}_j$ are known as unit dyads.

2. Algebra of Vectors and Tensors

2.1 Inner product or dot product of two vectors

$$\vec{A} = A_i \hat{e}_i$$
$$\vec{B} = B_j \hat{e}_j$$
$$\vec{A} \cdot \vec{B} = A_i B_j (\hat{e}_i \cdot \hat{e}_j)$$

For orthogonal unit vectors

$(\hat{e}_i \cdot \hat{e}_j) = \delta_{ij}$

where $\begin{matrix} \delta_{ij} = 1, & i = j \\ \delta_{ij} = 0, & i \neq j \end{matrix}$

$$\vec{A} \cdot \vec{B} = A_i B_j \delta_{ij} = A_i B_i \qquad \text{M.6}$$

2.2 Inner product (dot product) of a vector with a tensor of rank 2

$\vec{A} = A_k \hat{e}_k, \quad \overline{\overline{T}} = T_{ij} \hat{e}_i \hat{e}_j$

$$\begin{aligned} \vec{A} \cdot \overline{\overline{T}} &= A_k T_{ij} \left(\hat{e}_k \cdot \hat{e}_i\right) \hat{e}_j \\ \overline{\overline{T}} \cdot \vec{A} &= A_k T_{ij} \left(\hat{e}_j \cdot \hat{e}_k\right) \hat{e}_i \end{aligned} \qquad \text{M.7}$$

In general $\vec{A} \cdot \overline{\overline{T}} \neq \overline{\overline{T}} \cdot \vec{A}$

The equality <u>exists</u> only if $T_{ij} = T_{ji}$ i.e. tensor is <u>symmetric</u>

2.3 Outer or Cross product of two vectors

For an orthogonal set of unit vectors such that $\hat{e}_i \times \hat{e}_j = \hat{e}_k$ for cyclic i, j, k using either LH or RH rule

$$\vec{A} \times \vec{B} = e_{ijk} A_j B_k \hat{e}_i \qquad \text{M.8}$$

where e_{ijk} is the <u>alternating symbol</u> defined as

$e_{ijk} =$ 0 if any two of (i, j, k) are same

= +1 if $\{i, j, k\}$ is a cyclic permutation of {1, 2, 3}

= -1 otherwise

Therefore an outer or cross product of two vectors is a vector defined as,

$\vec{C} = \vec{A} \times \vec{B}$

$C_i = A_j B_k - A_k B_j$, $i \neq j \neq k$ and (i, j, k) must be a cyclic combination M.9

2.4 Dyadic product

A dyadic product of two vectors is a tensor of rank two defined as,

$$\vec{A}\vec{B} = A_i B_j \hat{e}_i \hat{e}_j \qquad \text{M.10}$$

2.5 Tensor Contractions or dot products

A dot product or Contraction of two second rank tensors $\overline{\overline{P}}$ and $\overline{\overline{Q}}$ is a second rank tensor defined as,

$$\overline{\overline{P}} \bullet \overline{\overline{Q}} = P_{ij} Q_{rs} (\hat{e}_j \cdot \hat{e}_r) \hat{e}_i \hat{e}_s \qquad \text{M.11}$$

For Orthogonal unit vectors,

$$\overline{\overline{P}} \bullet \overline{\overline{Q}} = P_{ij} Q_{rs} \delta_{jr} \hat{e}_i \hat{e}_s = P_{ij} Q_{js} \hat{e}_i \hat{e}_s \qquad \text{M.12}$$

A double dot product or double contraction of two second rank tensors is a scalar defined as,

$$\overline{\overline{P}} : \overline{\overline{Q}} = P_{ij} Q_{rs} (\hat{e}_j \cdot \hat{e}_r)(\hat{e}_i \cdot \hat{e}_s) \qquad \text{M.13}$$

For orthogonal unit vectors,

$$\overline{\overline{P}} : \overline{\overline{Q}} = P_{ij} Q_{rs} \delta_{jr} \delta_{is} = P_{ij} Q_{ji} \qquad \text{M.14}$$

3. Calculus of scalar, vector and tensor functions

3.1 Gradient

For a scalar function $\phi(\vec{r}, t)$ the gradient represents the information regarding the instantaneous spatial rate of change along the coordinate directions. The gradient is conveniently defined through the del operator given as,

$$\nabla \equiv \hat{e}_i \frac{\partial}{\partial s_i}, \qquad \text{M.15}$$

where s_i are related to the generalized coordinates x_i as

$$ds_i = h_i dx_i .$$

Physically ds_i represents the components along the coordinate direction of the differential line element $\vec{dl}$ i.e. $\vec{dl} = ds_i \hat{e}_i$

$$\nabla \equiv \hat{e}_i \frac{\partial}{h_i \partial x_i} \qquad \text{M.16}$$

The gradient of a scalar function ϕ is defined as,

$$\nabla \phi \equiv \hat{e}_i \frac{\partial \phi}{h_i \partial x_i} \qquad \text{M.17}$$

The gradient of a scalar is a vector.

The most important use of gradient of a scalar function is in finding the instantaneous spatial rate of change of ϕ along a <u>specified direction</u> characterized by a unit vector $\hat{m}$ as,

$$\frac{\partial\phi}{\partial m} = \nabla\phi\cdot\hat{m} \qquad \text{M.18}$$

The other important use is in finding the direction of the local normal to a given surface. A surface in 3D/2D can be expressed as a scalar function relation

$$\phi(\vec{r}) = Const.$$

Ex. $x^2 + y^2 + z^2 = 1.0$ spherical surface

The direction of local normal to the surface is given by the vector $\nabla\phi$ and therefore the local unit normal is given as,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \qquad \text{M.19}$$

3.2 Divergence, Curl and Gradient of a vector

The divergence operation is defined as,

$$\nabla\cdot\vec{A} = \left(\hat{e}_i \frac{\partial}{h_i\partial x_i}\right)\cdot\left(A_j\hat{e}_j\right) \text{ or,}$$

$$\nabla\cdot\vec{A} = \hat{e}_i\cdot\left(\frac{\partial(A_j\hat{e}_j)}{h_i\partial x_i}\right) \text{ or,}$$

$$\nabla\cdot\vec{A} = \frac{1}{h_i}\frac{\partial A_j}{\partial x_i}\left(\hat{e}_i\cdot\hat{e}_j\right) + A_j\left(\hat{e}_i\cdot\frac{\partial\hat{e}_j}{h_i\partial x_i}\right) \qquad \text{M.20}$$

For Orthogonal coordinate systems,

$$\nabla\cdot\vec{A} = \frac{1}{h_i}\frac{\partial A_i}{\partial x_i} + A_j\left(\hat{e}_i\cdot\frac{\partial\hat{e}_j}{h_i\partial x_i}\right) \qquad \text{M.21}$$

For Cartesian coordinates $h_i = 1.0$, $\frac{\partial\hat{e}_j}{\partial x_i} = 0$

$$\nabla\cdot\vec{A} = \frac{\partial A_i}{\partial x_i}$$

The Curl operation is defined as,

$$\nabla\times\vec{A}=\left(\hat{e}_j\frac{\partial}{h_j\partial x_j}\right)\times\left(A_k\hat{e}_k\right)\quad \text{or,}$$

$$\nabla\times\vec{A}=\hat{e}_j\times\left(\frac{\partial\left(A_k\hat{e}_k\right)}{h_j\partial x_j}\right)\quad \text{or,}$$

$$\nabla\times\vec{A}=\frac{1}{h_j}\frac{\partial A_k}{\partial x_j}\left(\hat{e}_j\times\hat{e}_k\right)+A_k\left(\hat{e}_j\times\frac{\partial\hat{e}_k}{h_j\partial x_j}\right) \tag{M.22}$$

For orthogonal coordinate systems,

$$\nabla\times\vec{A}=e_{ijk}\frac{1}{h_j}\frac{\partial A_k}{\partial x_j}\hat{e}_i+A_k\left(\hat{e}_j\times\frac{\partial\hat{e}_k}{h_j\partial x_j}\right) \tag{M.23}$$

For Cartesian coordinates,

$$\nabla\times\vec{A}=\left(\frac{\partial A_k}{\partial x_j}-\frac{\partial A_j}{\partial x_k}\right)\hat{e}_i$$

Gradient of vector which is a tensor of rank 2 is defined as,

$$\nabla\vec{A}=\left(\hat{e}_i\frac{\partial}{h_i\partial x_i}\right)\left(A_j\hat{e}_j\right)\quad \text{or,}$$

$$\nabla\vec{A}=\hat{e}_i\left(\frac{\partial A_j}{h_i\partial x_i}\hat{e}_j+A_j\frac{\partial\hat{e}_j}{h_i\partial x_i}\right)\quad \text{or,}$$

$$\nabla\vec{A}=\frac{1}{h_i}\frac{\partial A_j}{\partial x_i}\hat{e}_i\hat{e}_j+A_j\hat{e}_i\left(\frac{\partial\hat{e}_j}{h_i\partial x_i}\right)=g_{ij}\hat{e}_i\hat{e}_j \tag{M.24}$$

Another gradient of a vector that is useful in decomposing the above tensor into symmetric and anti-symmetric components is defined as,

$$\nabla\vec{A}^T=g_{ji}\hat{e}_i\hat{e}_j\quad .$$

For Cartesian coordinates, the two gradients are given as,

$$\nabla\vec{A}=\frac{\partial A_j}{\partial x_i}\hat{e}_i\hat{e}_j=g_{ij}\hat{e}_i\hat{e}_j\Rightarrow g_{ij}=\frac{\partial A_j}{\partial x_i},\quad \nabla\vec{A}^T=g_{ji}\hat{e}_i\hat{e}_j=\frac{\partial A_i}{\partial x_j}\hat{e}_i\hat{e}_j \tag{M.25}$$

3.3 Divergence of a Tensor of Rank 2

The divergence of a second rank tensor can be defined as,

$$\nabla \cdot \overline{\overline{T}} = \left(\hat{e}_i \frac{\partial}{h_i \partial x_i} \right) \cdot \left(T_{jk} \hat{e}_j \hat{e}_k \right) \quad \text{or,}$$

$$\nabla \cdot \overline{\overline{T}} = \left(\frac{\partial T_{jk}}{h_i \partial x_i} \right) \left(\hat{e}_i \cdot \hat{e}_j \right) \hat{e}_k + T_{jk} \left(\hat{e}_i \cdot \frac{\partial \hat{e}_j}{h_i \partial x_i} \right) \hat{e}_k + T_{jk} \left(\hat{e}_i \cdot \hat{e}_j \right) \frac{\partial \hat{e}_k}{h_i \partial x_i} \qquad \text{M.26}$$

For orthogonal systems,

$$\nabla \cdot \overline{\overline{T}} = \left(\frac{\partial T_{ik}}{h_i \partial x_i} \right) \hat{e}_k + T_{jk} \left(\hat{e}_i \cdot \frac{\partial \hat{e}_j}{h_i \partial x_i} \right) \hat{e}_k + T_{ik} \frac{\partial \hat{e}_k}{h_i \partial x_i} \qquad \text{M.27}$$

For Cartesian system,

$$\nabla \cdot \overline{\overline{T}} = \left(\frac{\partial T_{ik}}{\partial x_i} \right) \hat{e}_k \qquad \text{M.28}$$

The divergence of the dot product of a rank 2 tensor with a vector can be expressed as,

$$\nabla \cdot \overline{\overline{T}} \cdot \vec{A} = \left(\hat{e}_i \frac{\partial}{h_i \partial x_i} \right) \cdot \left[T_{jk} A_r \hat{e}_j \left(\hat{e}_k \cdot \hat{e}_r \right) \right] \quad \text{or,}$$

$$\nabla \cdot \overline{\overline{T}} \cdot \vec{A} = A_r \left(\hat{e}_k \cdot \hat{e}_r \right) \left(\frac{\partial T_{jk}}{h_i \partial x_i} \right) \left(\hat{e}_i \cdot \hat{e}_j \right) + T_{jk} \left(\frac{\partial \left(A_r \left(\hat{e}_k \cdot \hat{e}_r \right) \right)}{h_i \partial x_i} \right) \left(\hat{e}_i \cdot \hat{e}_j \right) + T_{jk} A_r \left(\hat{e}_k \cdot \hat{e}_r \right) \left(\hat{e}_i \cdot \frac{\partial \hat{e}_j}{h_i \partial x_i} \right)$$

M.29

For orthogonal systems,

$$\nabla \cdot \overline{\overline{T}} \cdot \vec{A} = A_k \left(\frac{\partial T_{ik}}{h_i \partial x_i} \right) + T_{ik} \left(\frac{\partial A_k}{h_i \partial x_i} \right) + T_{jk} A_k \left(\hat{e}_i \cdot \frac{\partial \hat{e}_j}{h_i \partial x_i} \right) \qquad \text{M.30}$$

For Cartesian systems,

$$\nabla \cdot \overline{\overline{T}} \cdot \vec{A} = A_k \left(\frac{\partial T_{ik}}{h_i \partial x_i} \right) + T_{ik} \left(\frac{\partial A_k}{h_i \partial x_i} \right) \qquad \text{M.31}$$

Calculus of Scalar, Vector and Tensor functions-Part 1

Learning objectives

Learning the various calculus operations in generalized coordinates

Learning about the del operator

Gradient of Scalar

Curl and Gradient of a Vector

Calculus of Scaler, Vectors & Tensors

The Calculus operations generally employed in the subject of Fluid mechanics :

1. Derivatives (partial) → $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial x_i}$, $\frac{D}{Dt}$, $\nabla \equiv$ Gradient operator

special → $\frac{D}{Dt}$

$$\frac{D\phi}{Dt} = \left[\frac{\partial \phi}{\partial t} + \vec{V} \cdot \nabla \phi\right]$$

ϕ can be a scaler like density, temperature, pressure or a vector like $\vec{V}$ or $\rho\vec{V}$

$\nabla\phi$, $\nabla \cdot \vec{V}$, $\nabla \cdot \overline{\overline{T}}$, $\nabla \times \vec{V}$ ⟶ Usage of ∇ operator

2. Surface and Volume Integrals

$-\oiint_S \vec{V} \cdot \hat{n}\, ds$, $\iint_S \rho\vec{V} \cdot \hat{n}\, ds$, $\oiint_S \hat{n} \cdot \overline{\overline{T}}\, ds$

$\iiint_{Vol} \rho \vec{f}_b\, d\forall$

The ~~Gradient~~ del Operator : ∇

The del operator ∇ can be used ~~to~~ defines many types of calculus operations that can be used to represent or express quantities of relevance in the subject of Fluid Mechanics.

For, e.g. $-\nabla p$ ⟶ pressure force/vol. acting on fluid particles

$\nabla \cdot \vec{V}$ ⟶ Vol. strain rate

The ~~gr~~ del operator can be written as,

$$\nabla \equiv \hat{e}_i \frac{\partial}{h_i \partial x_i}$$ (Summation over i is implied)

This a general defn in any coordinate system.

What is h_i & $\frac{\partial}{\partial x_i}$

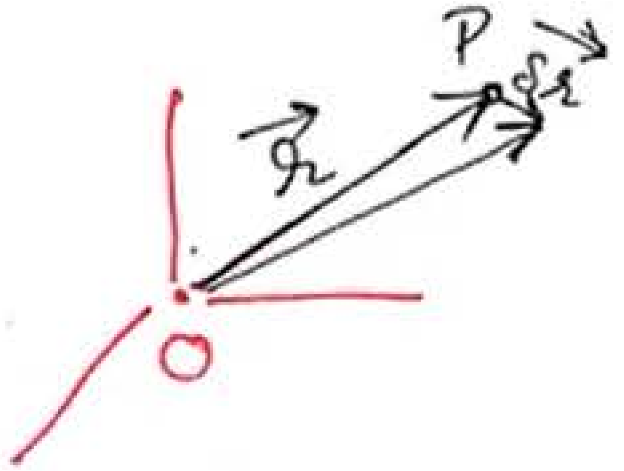
$\frac{\partial}{\partial x_i} \equiv$ partial derivative w.r.t coordinate x_i

$h_i \equiv$ scale factor for displacement along x_i

$\nabla \longrightarrow$ "It gives the spatial instantaneous rates of change of a quantity along the coordinate directions in the ~~neg~~ neighborhood of a point".

Why use scale factors h_i ??

Consider a point and its position vector $\vec{r}$ in a general coordinate system.



$$\vec{r} = x_i \hat{e}_i$$

$$\delta\vec{r} = \delta(x_i \hat{e}_i)$$

$$= (\delta x_i)\hat{e}_i + x_i \,\delta\hat{e}_i$$

$$\delta\hat{e}_i = \frac{\partial \hat{e}_i}{\partial x_j}\delta x_j \quad \text{(summation over } j \text{ is implied)}$$

$$\delta\vec{r} = \delta x_i \hat{e}_i + x_i \underbrace{\frac{\partial \hat{e}_i}{\partial x_j}}\delta x_j$$

$$\delta\vec{r} = \underbrace{h_i \,\delta x_i}_{\text{changes in coordinates}} \hat{e}_i$$

$$\delta\vec{r} = \underbrace{h_i \,\delta x_i}_{\Downarrow} \hat{e}_i$$

magnitude of displacement components along coordinate directions

In general the magnitude $\neq \delta x_i$

In fact there may be scaling factors involved 'h_i'.

Examples

Cartesian $\longrightarrow \delta\vec{r} = \delta x_1 \hat{e}_1 + \delta x_2 \hat{e}_2 + \delta x_3 \hat{e}_3$

$$= \delta x \,\hat{i} + \delta y \,\hat{j} + \delta z \,\hat{k}$$

$$\because \frac{\partial \hat{e}_i}{\partial x_j} = 0$$

$$\Rightarrow \boxed{\delta r = \delta x_i \hat{e}_i} \rightarrow \underline{h_i = 1.0}$$

~~Cylindrical $\longrightarrow \delta\vec{r} = \delta x_i \hat{e}_i$ $r\hat{e}_r + z\hat{e}_z$~~

All scaling factors are 1.0

Cylindrical

$\vec{r} = r\hat{e}_r + z\hat{e}_z$ → does not have any θ-comp.

$\vec{\delta r} = \delta(r\hat{e}_r) + \delta(z\hat{e}_z)$

$= \delta r\,\hat{e}_r + r\,\underbrace{\delta\hat{e}_r}_{\text{changes with } \theta.} + \delta z\,\underbrace{\hat{e}_z}_{\text{does not change.}}$

$$\delta\hat{e}_r = \frac{d\hat{e}_r}{d\theta}\,\delta\theta = \hat{e}_\theta\,\delta\theta$$

scale factors

$$\vec{\delta r} = 1(\delta r)\hat{e}_r + r\,\delta\theta\,\hat{e}_\theta + 1\,\delta z\,\hat{e}_z$$

Scale factor

changes in coordinates

Using indicial notation.

$\Longrightarrow \vec{\delta r} = h_i\,\delta x_i\,\hat{e}_i, \quad i = 1, 2, 3 = r, \theta, z$

$$\left.\begin{aligned} h_1 &= h_r = 1.0 \\ h_2 &= h_\theta = r \\ h_3 &= h_z = 1.0 \end{aligned}\right\}$$

Once scaling factors for a coordinate system are identified, we can write the del operator

$$\nabla \equiv \hat{e}_i \frac{\partial}{h_i\,\partial x_i} \xrightarrow{\text{Cartesian}} \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

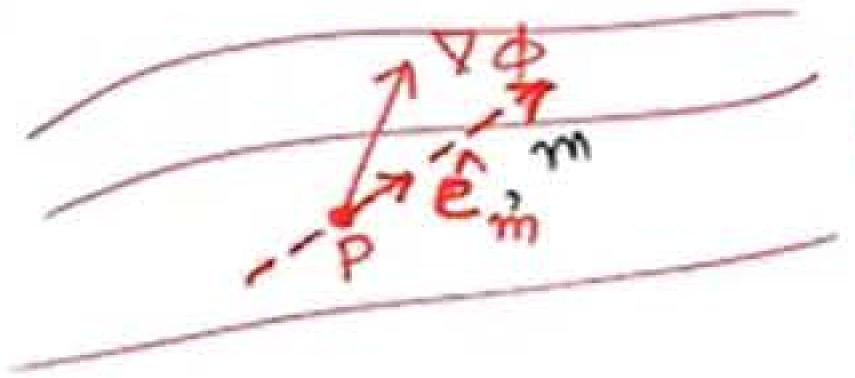
$$\xrightarrow{\text{Cylindrical}} \hat{e}_r\frac{\partial}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial}{\partial\theta} + \hat{e}_z\frac{\partial}{\partial z}$$

Using the del operator.

1. Gradient of a scalar.

$$\nabla\phi \equiv \hat{e}_i \frac{\partial\phi}{h_i\,\partial x_i} \rightarrow \text{vector}$$

Use : To find rate of change of ϕ along a given direction



$$\left(\frac{d\phi}{dm}\right)_P = (\nabla\phi)_P \cdot \hat{e}_m$$

Another use in geometry is to find the unit normal vector to a surface $\phi(\vec{r}) = 0$

e.g. Spherical surface, $\underbrace{x^2+y^2+z^2-R^2}_{\phi(x,y,z)} = 0$

paraboloidal surface

$$z = x^2+y^2$$

or $\underbrace{x^2+y^2-z}_{\phi(x,y,z)} = 0$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \rightarrow \text{unit normal}$$

Curl and Gradient of a Vector

$$\nabla \times \vec{A} \longrightarrow \text{Curl}(\vec{A})$$

$$\nabla \times \vec{A} = \left(\hat{e}_i \frac{\partial}{h_i \partial x_i}\right) \times (A_j \hat{e}_j)$$

$$= \hat{e}_i \times \left[\frac{\partial A_j}{h_i \partial x_i} \hat{e}_j + A_j \frac{\partial \hat{e}_j}{h_i \partial x_i}\right]$$

$$\boxed{\nabla \times \vec{A} = \frac{\partial A_j}{h_i \partial x_i} (\hat{e}_i \times \hat{e}_j) + A_j \, \hat{e}_i \times \frac{\partial \hat{e}_j}{h_i \partial x_i}}$$

↳ for a general coordinate system.

For orthogonal systems

$$\nabla \times \vec{A} = \frac{\partial A_j}{h_i \partial x_i} e_{ijk} \hat{e}_k + A_j \, \hat{e}_i \times \frac{\partial \hat{e}_j}{h_i \partial x_i}$$

e_{ijk} ⇓ permutation symbol.

- +1 cyclic (i,j,k)
- −1 anticyclic (i,j,k)
- 0 otherwise

Cartesian System

$$\nabla \times \vec{A} = \frac{\partial A_j}{h_i \partial x_i} e_{ijk} \hat{e}_k \longrightarrow \text{Sum over } i, j, k$$

But due to properties of permutation symbol e_{ijk}, we have only six non-zero terms.

$$\cancel{e_{123}} \; \cancel{e_{213}} \; e_{123} = e_{231} = e_{312} = +1$$

$$e_{213} = e_{321} = e_{132} = -1$$

$$\boxed{\nabla \times \vec{A} = \left(\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}\right)\hat{e}_3 + \left(\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}\right)\hat{e}_1 + \left(\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}\right)\hat{e}_2}$$

↳ Cartesian Coordinates

$x_1 = x$, $\hat{e}_1 = \hat{i}$

$x_2 = y$, $\hat{e}_2 = \hat{j}$

$x_3 = z$, $\hat{e}_3 = \hat{k}$

$A_1 = A_x$

$A_2 = A_y$

$A_3 = A_z$

The above result is useful for expressing vorticity vector in 3D

$$\boxed{\vec{\Omega} = \nabla \times \vec{V}}$$

↳ general expression for vorticity

Cylindrical Coordinates

$$\nabla \times \vec{A} = \underbrace{\frac{\partial A_j}{h_i \partial x_i} (\hat{e}_i \times \hat{e}_j)}_{\text{Term I}} + \underbrace{A_j \, \hat{e}_i \times \frac{\partial \hat{e}_j}{h_i \partial x_i}}_{\text{Term II}}$$

$$\text{Term I} = \frac{\partial A_\theta}{\partial r}(\hat{e}_r \times \hat{e}_\theta) + \frac{\partial A_z}{\partial r}(\hat{e}_r \times \hat{e}_z)$$
$$+ \frac{\partial A_r}{r\,\partial\theta}(\hat{e}_\theta \times \hat{e}_r) + \frac{\partial A_z}{r\,\partial\theta}(\hat{e}_\theta \times \hat{e}_z)$$
$$+ \frac{\partial A_r}{\partial z}(\hat{e}_z \times \hat{e}_r) + \frac{\partial A_\theta}{\partial z}(\hat{e}_z \times \hat{e}_\theta)$$

Now $\hat{e}_r \times \hat{e}_\theta = \hat{e}_z$

$\hat{e}_\theta \times \hat{e}_z = \hat{e}_r$

$\hat{e}_z \times \hat{e}_r = \hat{e}_\theta$

$$\text{Term - I} = \left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)\hat{e}_z + \left(\frac{\partial A_z}{r\,\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\theta$$

$$\text{Term - II} = A_j\,\hat{e}_i \times \frac{\partial \hat{e}_j}{h_i\,\partial x_i}$$

$$= A_j\,\hat{e}_\theta \times \frac{\partial \hat{e}_j}{r\,\partial\theta} = \frac{A_r}{r}(\hat{e}_\theta \times \hat{e}_\theta) \xrightarrow{} 0 + \frac{A_\theta}{r}\,\hat{e}_\theta \times \frac{\partial \hat{e}_\theta}{\partial\theta}$$

$$\text{Term-II} = +\frac{A_\theta}{r}\,\hat{e}_z$$

$$\boxed{\nabla \times \vec{A} = \left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta} + \frac{A_\theta}{r}\right)\hat{e}_z + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial A_z}{r\,\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)\hat{e}_r}$$

Gradient of Vector

$$\nabla\vec{A} = \left(\hat{e}_i \frac{\partial}{h_i\,\partial x_i}\right)\left(A_j\,\hat{e}_j\right)$$

$$= \hat{e}_i\left(\underbrace{\frac{\partial A_j}{h_i\,\partial x_i}\hat{e}_j}_{\text{Dyadic product}} + A_j\frac{\partial \hat{e}_j}{h_i\,\partial x_i}\right)$$

$$= \hat{e}_i\frac{\partial A_j}{h_i\,\partial x_i}\hat{e}_j + \hat{e}_i\,A_j\frac{\partial \hat{e}_j}{h_i\,\partial x_i}$$

$$= \frac{\partial A_j}{h_i\,\partial x_i}\hat{e}_i\hat{e}_j + A_j\,\hat{e}_i\underbrace{\frac{\partial \hat{e}_j}{h_i\,\partial x_i}}_{\text{vector}}$$

Dyadic product

RHS is a tensor of rank 2

$\nabla\vec{A} \longrightarrow$ is a tensor of rank 2.

For Cartesian coordinates, $h_i = 1.0$

$$\frac{\partial \hat{e}_j}{\partial x_i} = 0$$

$$\boxed{\nabla\vec{A} = \frac{\partial A_j}{\partial x_i}\hat{e}_i\hat{e}_j}$$

In general gradient of a vector is a tensor of rank 2.

$$\Rightarrow \quad \nabla\vec{A} = g_{ij}\,\hat{e}_i\,\hat{e}_j$$

There is another related tensor to $\nabla\vec{A}$ that helps in expressing local strain rates in 3D. This is ~~defined~~ defined as,

$$\nabla\vec{A}^{T} = g_{ji}\,\hat{e}_i\,\hat{e}_j \longrightarrow \text{read as transpose } \nabla\vec{A}$$

when g_{ij} are components of $\nabla\vec{A}$.

For example.

In cartesian $\nabla\vec{A} = \underbrace{\dfrac{\partial A_j}{\partial x_i}}_{=\,g_{ij}}\,\hat{e}_i\,\hat{e}_j$

$$\Rightarrow \nabla\vec{A}^{T} = \underbrace{\frac{\partial A_i}{\partial x_j}}_{=\,g_{ji}}\,\hat{e}_i\,\hat{e}_j$$

For cylindrical, $\longrightarrow$ Starting from general relation

$$\nabla\vec{A} = \frac{\partial A_j}{h_i\,\partial x_i}\,\hat{e}_i\,\hat{e}_j + A_j\,\hat{e}_i\,\frac{\partial \hat{e}_j}{h_i\,\partial x_i}$$

we obtain ~~expand~~ expansion of right hand side and obtain ~~the~~ componets g_{ij}. i.e $g_{rr}, g_{r\theta}, g_{rz}, \ldots\ldots$.

In order ~~to~~ test your understanding ~~this~~ is left as an exercise.

Now with this background. we can expres ~~the~~ local strain rates and rotation rates in coordinate free-forms valid for both 2D / 3D flow fields.

x ———————— x.

Vorticity Vector and Strain Rate Tensor

Learning Objectives

To express Vorticity vector in a coordinate free-form and examine some of its properties

Identify the Strain rate tensor and relate it to the local deformation rates

Examine properties of Strain rate tensor

Strain rate Tensor & Vorticity

It has been shown from first principles for a 2D flow, how we can visualize and express local deformation and rotation rates through material line (infinitesimal) kinematics. While it is not very difficult to extend ~~their~~ expressions for a Cartesian coordinate system in 3D flow domain, we adopt a coordinate-free approach.

$$\boxed{\vec{\Omega} = \nabla \times \vec{V}} \rightarrow \text{vorticity vector}$$

1. The above result is valid for any coordinate system and for 2D/3D flows.

$$\vec{\Omega} = \left(\hat{e}_i \frac{\partial}{h_i \partial x_i}\right) \times \left(v_j \hat{e}_j\right)$$

$$= \frac{\partial v_j}{h_i \partial x_i}\left(\hat{e}_i \times \hat{e}_j\right) + v_j \hat{e}_i \times \frac{\partial \hat{e}_j}{h_i \partial x_i}$$

For Cartesian

$$\vec{\Omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \underbrace{\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{\text{derived earlier in 2D using first principles}}\hat{k}$$

$$\vec{V} \equiv (u, v, w)$$

In Cylindrical coordinates

$$\vec{V} \equiv (v_r, v_\theta, v_z)$$

$$\vec{\Omega} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\hat{e}_z$$

Remarks:

1. For 2D flows:

$\boxed{\text{Cartesian:}}$ $\vec{V} \equiv (u, v)$, $u = f(x, y, t)$, $v = func(x, y, t)$

$w = 0$, $\dfrac{\partial u}{\partial z} = \dfrac{\partial v}{\partial z} = 0$

$$\vec{\Omega} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k} \Rightarrow$$ Vorticity vector is $\perp$ to the planar flow domain

$\boxed{\text{Cylindrical:}}$ a) $\vec{V} \equiv (V_r, V_\theta) \Rightarrow V_z = 0, \ \frac{\partial()}{\partial z} = 0$

b) Axi symmetric flow

$\vec{V} \equiv (V_r, V_z), \ V_\theta = 0, \ \frac{\partial()}{\partial \theta} = 0$

a) $\vec{\Omega} = \underbrace{\left(\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r}\frac{\partial V_r}{\partial \theta} \right) \hat{e}_z}$

↳ $\perp$ to 2D flow domain i.e. $(r-\theta)$ plane

b) $\vec{\Omega} = \underbrace{\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} \right) \hat{e}_\theta}$

↳ $\perp$ to 2D flow domain i.e $(r-z)$ plane

All 2D flows $\Rightarrow$ vorticity vector has a single component $\perp$ ~~to the~~ planar flow domain.

② Vorticity field is ~~and~~ unconditionally divergence free.

We can show that $\nabla \cdot \vec{\Omega} = 0$ at all points and at all instants of time.

Proof: Consider the identity

$\nabla \cdot (\nabla \times \vec{A}) = 0$ (Sheet of Identities, theorems from vector calculus)

Take $\vec{V} \equiv \vec{A}$

$\Rightarrow \nabla \cdot (\nabla \times \vec{V}) = 0$

or $\boxed{\nabla \cdot \vec{\Omega} = 0}$

The above result can be interpreted in another way.

Using Gauss theorem

$$\iiint_{Vol} (\nabla \cdot \vec{\Omega}) \, d\forall = \oiint_{Surf} \vec{\Omega} \cdot \hat{n} \, ds$$

$$= 0$$

Nett flux of vorticity through a closed surface is zero.

This ~~results~~ result reminds of the similar property exhibited ~~shared~~ by magnetic field $\vec{B}$. [$\nabla\cdot\vec{B} = 0 \rightarrow$ Maxwell's equation]

x ——————— x.

Strain rate tensor

$$\bar{\bar{S}} = \left(\frac{\nabla\vec{V} + \nabla\vec{V}^T}{2}\right) \longrightarrow \text{Coordinate free form.}$$

We have already discussed the $\nabla\vec{A}$, $\nabla\vec{A}^T$ gradients of a vector.

let $\nabla\vec{V} = g_{ij}\,\hat{e}_i\hat{e}_j$

Then, $\nabla\vec{V}^T = g_{ji}\,\hat{e}_i\hat{e}_j$

$$\bar{\bar{S}} = \left(\frac{g_{ij} + g_{ji}}{2}\right)\hat{e}_i\hat{e}_j$$

$$\nabla\vec{V} = \frac{\partial v_j}{h_i\partial x_i}\hat{e}_i\hat{e}_j + v_j\hat{e}_i\frac{\partial\hat{e}_j}{h_i\partial x_i}$$

Cartesian Coordinates

$$\nabla\vec{V} = \frac{\partial v_j}{h_i\partial x_i}\hat{e}_i\hat{e}_j = g_{ij}\hat{e}_i\hat{e}_j$$

$\searrow 1$, 0

$$g_{ij} = \frac{\partial v_j}{\partial x_i}, \quad g_{ji} = \frac{\partial v_i}{\partial x_j}$$

$$\bar{\bar{S}} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)\hat{e}_i\hat{e}_j$$

$$= S_{ij}\,\hat{e}_i\hat{e}_j$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right) \rightarrow \text{Components of strain rate tensor}$$

$$\Rightarrow S_{11} = \frac{\partial v_1}{\partial x_1}, \quad S_{22} = \frac{\partial v_2}{\partial x_2}$$

$$S_{33} = \frac{\partial v_3}{\partial x_3} \Rightarrow \left.\begin{matrix} S_{11} \\ S_{22} \\ S_{33}\end{matrix}\right\} \text{Three longitudinal strain rates}$$

$$S_{11} = \epsilon_{xx} = \frac{\partial u}{\partial x}$$

$$S_{22} = \epsilon_{yy} = \frac{\partial v}{\partial y}$$

$$S_{33} = \epsilon_{zz} = \frac{\partial w}{\partial z}$$

$$S_{12} = \frac{1}{2}\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right) = \frac{1}{2}\gamma_{12} = \frac{1}{2}\gamma_{12}$$

$$S_{23} = \frac{1}{2}\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right) = \frac{1}{2}\gamma_{23} = \frac{1}{2}\gamma_{32}$$

$$S_{13} = \frac{1}{2}\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right) = \frac{1}{2}\gamma_{13} = \frac{1}{2}\gamma_{31}$$

$$(S_{12},\ S_{23},\ S_{13}) \longrightarrow \frac{1}{2}(\text{shear strain rates})$$

We observe that $\bar{\bar{S}}$ is a symmetric tensor i.e $S_{ij} = S_{ji}$

~~Observe that~~, .

Another <u>observation</u>

$$S_{11} + S_{22} + S_{33} = \frac{\partial u_i}{\partial x_i} \rightarrow (\text{Sum over } i)$$

$\Rightarrow$ Vol. strain rate

$$\epsilon_V = \nabla\cdot\vec{V} = S_{11} + S_{22} + S_{33}.$$

Let us examine the Strain tensor in cylindrical coordinates

$$\nabla\vec{V} = \frac{\partial v_j}{h_i \partial x_i}\hat{e}_i\hat{e}_j + v_j \hat{e}_i \frac{\partial \hat{e}_j}{h_i \partial x_i}$$

Using the procedures already discussed the two terms on the right can be expanded to have the following result

$$\nabla\vec{V} = \frac{\partial V_r}{\partial r}\hat{e}_r\hat{e}_r + \frac{\partial V_\theta}{\partial r}\hat{e}_r\hat{e}_\theta + \frac{\partial V_z}{\partial r}\hat{e}_r\hat{e}_z +$$
$$\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} - \frac{V_\theta}{r}\right)\hat{e}_\theta\hat{e}_r + \left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}\right)\hat{e}_\theta\hat{e}_\theta + \frac{1}{r}\frac{\partial V_z}{\partial \theta}\hat{e}_\theta\hat{e}_z +$$
$$\frac{\partial V_r}{\partial z}\hat{e}_z\hat{e}_r + \frac{\partial V_\theta}{\partial z}\hat{e}_z\hat{e}_\theta + \frac{\partial V_z}{\partial z}\hat{e}_z\hat{e}_z$$
$$= g_{ij}\hat{e}_i\hat{e}_j$$

So, g_{ij} can be directly written by <u>inspection</u>.

Therefore, components of $\bar{\bar{S}}$ can be written as,

$$S_{ij} = \frac{g_{ij} + g_{ji}}{2}$$

$$S_{11} = \frac{\partial V_r}{\partial r}, \quad S_{22} = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{V_r}{r}, \quad S_{33} = \frac{\partial V_z}{\partial z}$$

$$\underset{r\ \theta}{S_{12}} = \underset{\theta\ r}{S_{21}} = \frac{1}{2}\left(\frac{\partial V_\theta}{\partial r} - \frac{V_\theta}{r} + \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right) = \frac{1}{2}\gamma_{r\theta} = \frac{1}{2}\gamma_{\theta r}$$

$$\underset{\theta\ z}{S_{23}} = S_{32} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial V_z}{\partial \theta} + \frac{\partial V_\theta}{\partial z}\right) = \frac{1}{2}\gamma_{\theta z} = \frac{1}{2}\gamma_{z\theta}$$

$$\underset{r\ z}{S_{13}} = S_{31} = \frac{1}{2}\left(\frac{\partial V_z}{\partial r} + \frac{\partial V_r}{\partial z}\right) = \frac{1}{2}\gamma_{rz} = \frac{1}{2}\gamma_{zr}$$

Again we observe

$$\boxed{\epsilon_V = \nabla\cdot\vec{V} = S_{11} + S_{22} + S_{33}}$$

It is left as an exercise to obtain the components of strain rate tensor for 2D flows
- Cartesian
- Cylindrical.

Finally to conclude, the strain rate tensor components can be arranged in a matrix as,

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

$\frac{1}{2}$ (shear strain rates)

Longitudinal strain rates

x ———————— x

Forces on a fluid particle

Learning Objectives

To develop a mathematical model for expressing forces on a fluid particle

To understand the physical meaning of divergence of stress tensor

Expressing forces acting on a fluid particle

- Body forces → Originate due to external force field
- Surface forces → They originate due to force interactions between a fluid particle and the surrounding fluid. (Internal action)

Examples of body forces

- Gravity → most common force field acting on all matter.

"Do not confuse b/w gravitational force between ~~ng~~ neighboring fluid particles" → This would be insignificant in comparison to Earth's gravitational field.

- Electrically charged fluid (plasma or even water with dissolved salts and ions) subjected to Electric and magnetic fields (external)

A fundamental difference between the two types of forces:

"The body forces are caused by external action of force fields that are generated independant of fluid motion but may be influenced by fluid motion".

- Gravitational field of earth → not influenced by fluid motion
- Lorentz force on electrically charged fluid in presence of external electric / magnetic fields → directly influenced by local fluid velocity and induced electric and magnetic fields

"The surface force on a fluid particle is dependent on velocity field and ~~are~~ has ~~have~~ microscopic origin related to force potentials and momentum exchange at the molecular level".

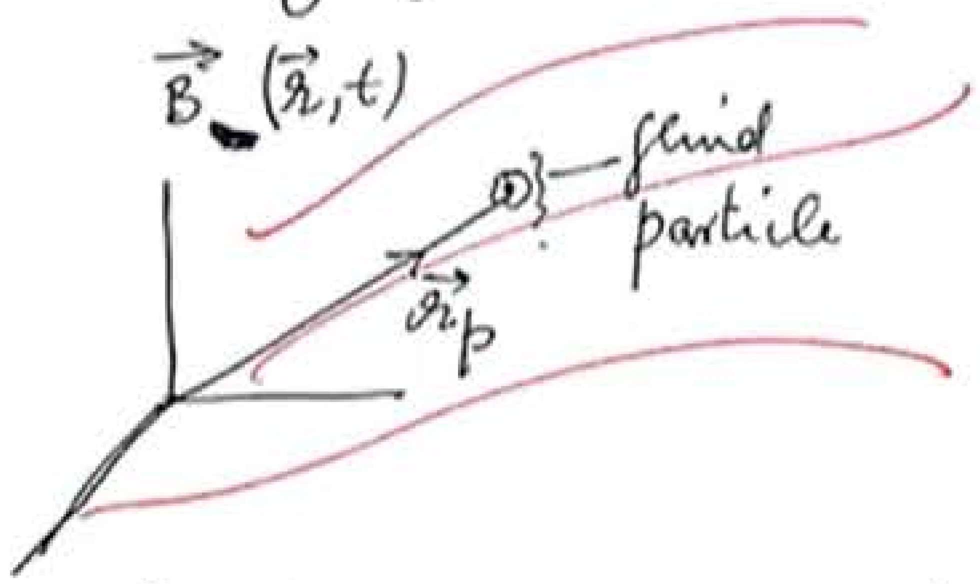
Forces on a fluid particle

body forces
origin
External force field

Surface forces
origin
force fields and momentum exchange at microscopic level.

Representation of forces

1. Body forces :



Let $\vec{B}(\vec{r},t)$ be the body force field intensity expressed either per unit mass or per unit vol.

The instantaneous body force on a fluid particle at $\vec{r}_p = \rho \underbrace{\vec{B}(\vec{r}_p,t)}_{\text{per unit mass}} d\forall$

If $\vec{B}(\vec{r}_p,t)$ is expressed per unit volume,

Body force $= \vec{f}_b = \vec{B}\, d\forall$

$$\boxed{\vec{f}_b = \vec{B}(\vec{r}_p,t)\, d\forall}$$

Ex. 1) gravitational field

$\vec{f}_b = \rho \underbrace{\vec{g}}_{\text{field intensity / mass.}} d\forall \longrightarrow$ force on fluid particle

2) Electric + Magnetic field

$\vec{f}_b = \underbrace{\sigma}_{\text{charge density / vol}}(\vec{E} + \vec{V} \times \vec{B})\, d\forall \longrightarrow$

$\vec{E} = \vec{E}_{ext} + \vec{E}_{induced}$

$\vec{B} = \vec{B}_{ext} + \vec{B}_{induced}$

Summary

$\vec{f}_b = \rho \vec{B}(\vec{r},t)\, d\forall$

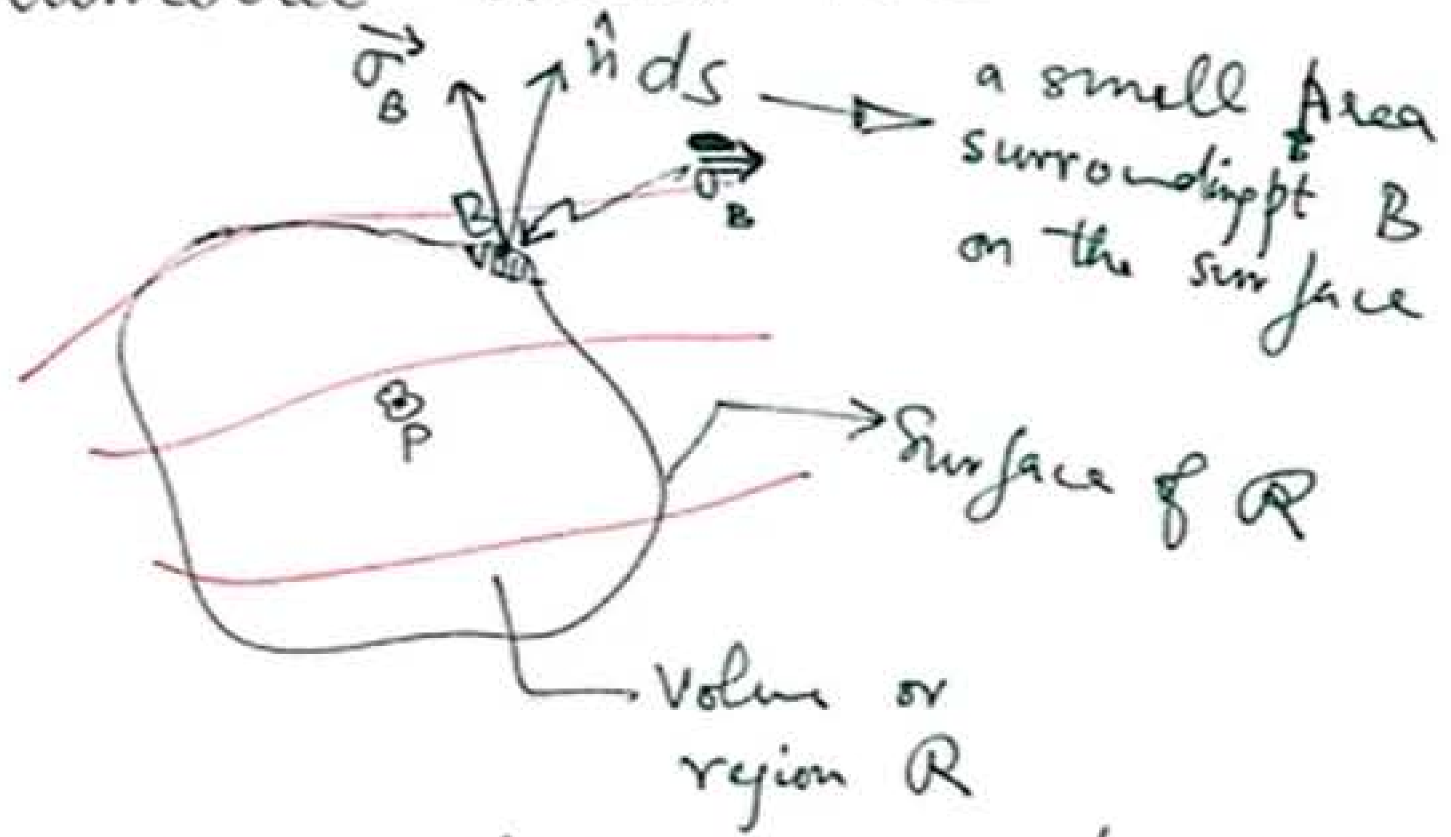
$= \rho(\vec{r},t)\underbrace{\vec{B}(\vec{r}_0,t)}_{\text{per unit mass.}} d\forall$

or

$f_b = \underbrace{\vec{B}(\vec{r},t)}_{\text{per unit vol.}} d\forall$

Surface force on a fluid particle

The approach used is similar to the one adopted in deriving the volumetric strain rate.



$\vec{\sigma}_B$: Stress vector at pt B on the area element surrounding B.

$$\vec{\sigma}_B = (\sigma_{n1}, \sigma_{n2}, \sigma_{n3})$$
$$= \sigma_{n1}\hat{e}_1 + \sigma_{n2}\hat{e}_2 + \sigma_{n3}\hat{e}_3$$

We have learnt in earlier courses on Solid / Fluid Mechanics, the stress vector on a infinitesimal planer area through a pt in the continuum can be expressed in terms of stress vector components along three coordinate planes passing through the same pt.

We know that,

$$\sigma_{n1} = n_1\sigma_{11} + n_2\sigma_{21} + n_3\sigma_{31} = \hat{n}\cdot(\sigma_{11}\hat{e}_1 + \sigma_{21}\hat{e}_2 + \sigma_{31}\hat{e}_3)$$
$$\sigma_{n2} = n_1\sigma_{12} + n_2\sigma_{22} + n_3\sigma_{32} = \hat{n}\cdot(\sigma_{12}\hat{e}_1 + \sigma_{22}\hat{e}_2 + \sigma_{32}\hat{e}_3)$$
$$\sigma_{n3} = n_1\sigma_{13} + n_2\sigma_{23} + n_3\sigma_{33} = \hat{n}\cdot(\sigma_{13}\hat{e}_1 + \sigma_{23}\hat{e}_2 + \sigma_{33}\hat{e}_3)$$

n_1, n_2, n_3 are components of unit vector $\hat{n}$ along the coordinate directions.

$$\vec{\sigma}_B = \sigma_{n1}\hat{e}_1 + \sigma_{n2}\hat{e}_2 + \sigma_{n3}\hat{e}_3$$
$$= \hat{n}\cdot(\sigma_{11}\hat{e}_1\hat{e}_1 + \sigma_{21}\hat{e}_2\hat{e}_1 + \sigma_{31}\hat{e}_3\hat{e}_1) + \hat{n}\cdot(\sigma_{12}\hat{e}_1\hat{e}_2 + \sigma_{22}\hat{e}_2\hat{e}_2 + \sigma_{32}\hat{e}_3\hat{e}_2) + \hat{n}\cdot(\sigma_{13}\hat{e}_1\hat{e}_3 + \sigma_{23}\hat{e}_2\hat{e}_3 + \sigma_{33}\hat{e}_3\hat{e}_3)$$

$$\vec{\sigma}_B = \hat{n}\cdot\left[\sigma_{ij}\hat{e}_i\hat{e}_j\right]_B = \hat{n}\cdot\bar{\bar{\sigma}}_B$$

$$\boxed{\vec{\sigma}_B = \hat{n}\cdot\bar{\bar{\sigma}}_B}$$ → Surface force intensity exerted by fluid on the side of the normal

* We know that
$$\hat{n}\cdot\bar{\bar{\sigma}} = \bar{\bar{\sigma}}\cdot\hat{n} \quad \text{for symmetric } \bar{\bar{\sigma}}$$
$$\Rightarrow \vec{\sigma}_B = \hat{n}\cdot\bar{\bar{\sigma}}_B = \bar{\bar{\sigma}}_B\cdot\hat{n}$$

Now we can express the force vector on the elemental surface area as,

$$d\vec{f}_s = \vec{\sigma}_B\, ds = (\hat{n}\cdot\bar{\bar{\sigma}}_B)\,ds = \underbrace{\bar{\bar{\sigma}}_B\cdot\hat{n}\,ds}_{\text{for symm. } \bar{\bar{\sigma}}}$$

Total or nett instantaneous surface force exerted ~~on~~ by surroundings on material region 'R' :

$$(\vec{f}_s)_R = \oiint (\hat{n}\cdot\bar{\bar{\sigma}})\,ds$$

To obtain the instantaneous surface force on a fluid particle inside region 'R'

$$\Rightarrow \quad \vec{f}_s = \lim_{\text{region } R \to d\forall} (\vec{f}_s)_R$$

$$= \lim_{\text{region } R \to d\forall} \left[\oiint \hat{n}\cdot\bar{\bar{\sigma}}\,ds\right]$$

By Gauss theorem

$$\oiint_S (\hat{n}\cdot\bar{\bar{\sigma}})\,ds = \iiint_R (\nabla\cdot\bar{\bar{\sigma}})\,d\forall$$

$$\vec{f}_s = \lim_{\text{region } R \to d\forall} \iiint (\nabla\cdot\bar{\bar{\sigma}})\,d\forall = (\nabla\cdot\bar{\bar{\sigma}})\,d\forall$$

$$\vec{f}_s / d\forall = \nabla\cdot\bar{\bar{\sigma}}$$

Surface force (instantaneous) at a point acting on a fluid particle per unit vol $= \nabla\cdot\bar{\bar{\sigma}} \rightarrow$ a vector

This is a very significant result.

The divergence of stress tensor at a point in the flow domain gives the resultant surface force ~~actin~~ per unit vol acting on a fluid particle instantaneously located at that point.

Summarize

Nett force on ~~the~~ a fluid particle at an instant

$$= \vec{f}_b + \vec{f}_s$$

$$\vec{f}_{nett} = \left(\rho(\vec{r},t)\,\vec{B}(\vec{r},t) + \nabla\cdot\bar{\bar{\sigma}}\right)d\forall$$

Governing equations of Fluid motion

Learning Objectives

Application of Physical laws or Principles : General approaches

Application of Law of Conservation of Mass to obtain the continuity equation (differential)

Governing Equations of Fluid Motion

By applying macroscopic, well known principles of physics (classical), to an individual fluid particle, we can obtain the basic equations that govern fluid motion.

Basic physical laws

→ a fluid particle

→ Infinitesimal C.V.

a fluid particle

1. Since a fluid particle is a smallest physical entity, "laws can be directly applied"
2. The working is short and we get the result very quickly

Infinitesimal C.V.

1. A C.V is not a physical entity (matter). Hence "laws cannot be applied directly"
2. Reynold's Transport theorem has to be employed with a choice of C.V.

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2. Reynold's Transport theorem has to be employed with a choice of C.V.

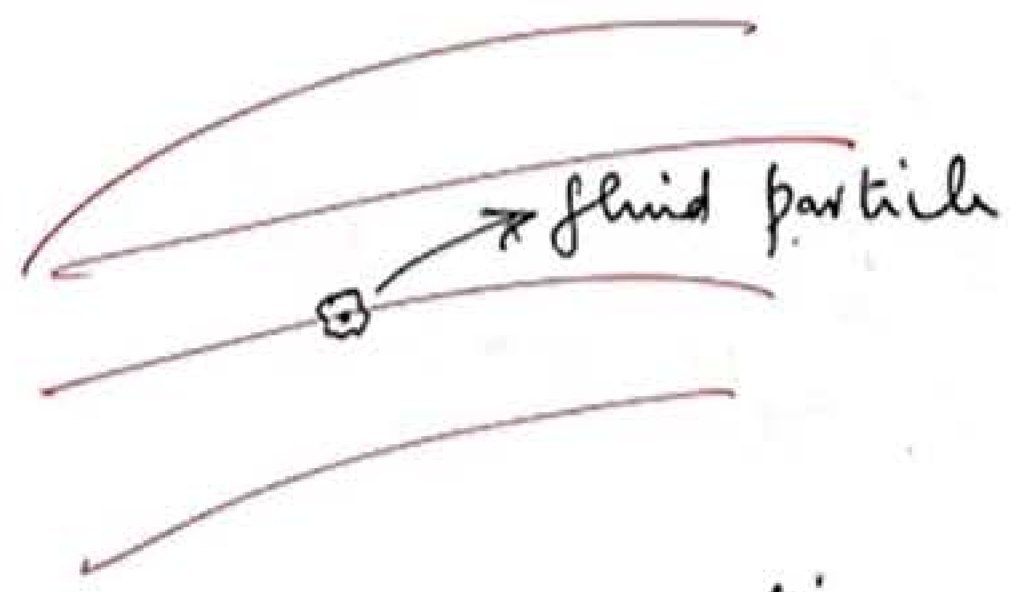
Basic laws $\xrightarrow[\text{CV}]{\text{infinitesimal}}$ $\boxed{\text{Conservative form of Equations}}$

$\Updownarrow$

Basic laws $\xrightarrow[\text{particle}]{\text{a fluid}}$ $\boxed{\text{Non-conservative form. of equations}}$

Once the equations are obtained, they are fully equivalent

Law of Mass Conservation



law of mass conservation states that "mass of a fluid particle must be preserved in time".

$$\Longrightarrow \frac{D(\text{mass})_{f.p}}{Dt} = 0$$

$$\frac{D}{Dt}(\text{density} \times \text{Vol}) = 0$$

$$\frac{D}{Dt}(\rho\,\delta\forall) = 0$$

Open using product rule,

$$\delta\forall \frac{D\rho}{Dt} + \rho \frac{D\delta\forall}{Dt} = 0$$

Dividing by $\rho\delta\forall$, we get

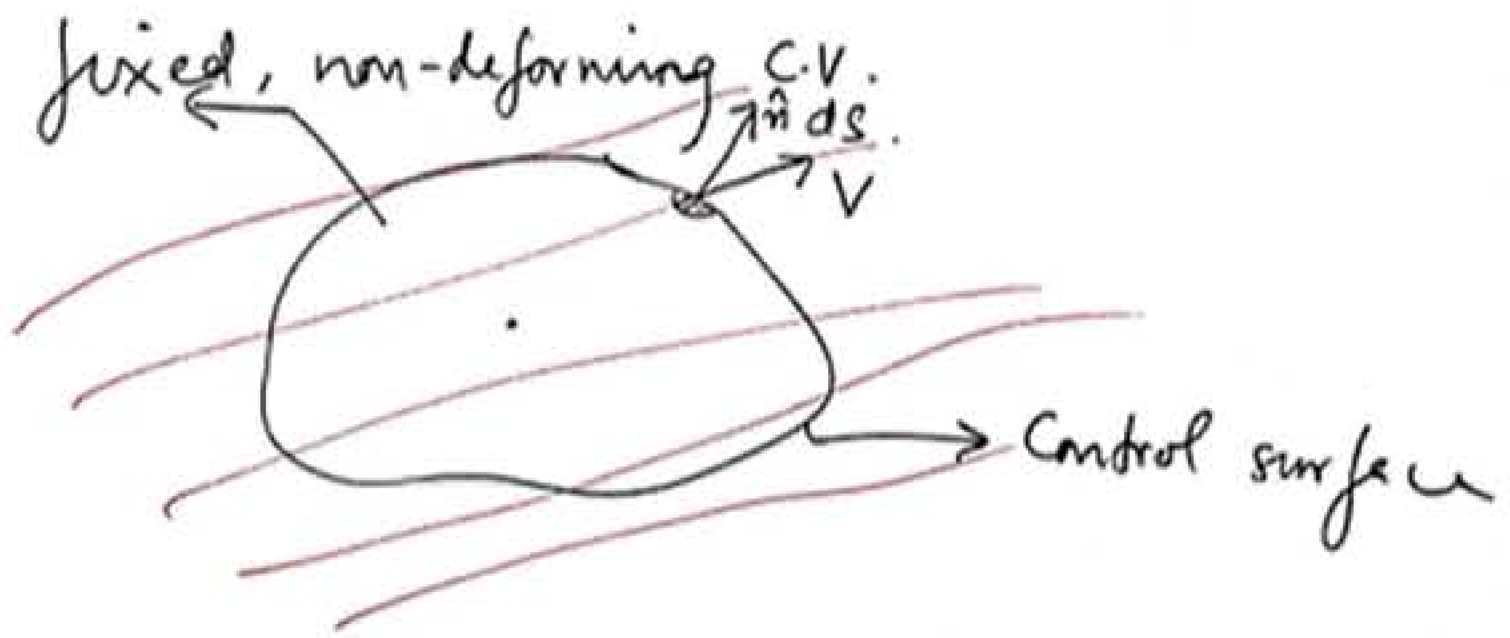
$$\frac{1}{\rho}\frac{D\rho}{Dt} + \underbrace{\frac{1}{\delta\forall}\frac{D(\delta\forall)}{Dt}}_{\text{vol. strain rate}} = 0$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0} \longrightarrow G.1$$

Continuity eqn. (non-conservative)

The above derivation clearly demonstrate the rather direct usage of a physical principle

We can also utilize the infinitesimal C.V approach and obtain the equation of continuity.



Using Reynolds Transport theorem and applying it for mass as the sys. property

$$\frac{Dm}{Dt} = 0 = \frac{\partial}{\partial t}\iiint_{C.V} \rho\, d\forall + \oiint \rho\vec{V}\cdot\hat{n}\, ds$$

Since C.V is rigid and non-deforming

$$0 = \iiint \frac{\partial\rho}{\partial t}\, d\forall + \oiint \rho\vec{V}\cdot\hat{n}\, ds$$

$\Downarrow$ Gauss theorem

$$0 = \iiint_{C.V} \frac{\partial\rho}{\partial t}\, d\forall + \iiint_{C.V} \nabla\cdot(\rho\vec{V})\, d\forall$$

$$0 = \iiint_{C.V.} \left[\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V}\right] d\forall$$

Since the integral on the R.H.S = 0 and the limits or Volume is arbitrary

$$\Rightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0} \rightarrow G.1'$$

This is also Continuity eqn.

Non-Conservative form.	Conservative form
$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0$	$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0$
$\Downarrow$ applies to a fluid particle that is moving	$\Downarrow$ applies to a tiny infinitesimal volume in space (fixed, non-deforming).

Both are equivalent statements of law of mass conservation.

Starting with G.1 $\longrightarrow$ G.1'

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{V} = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{V}\cdot\nabla\rho + \rho\nabla\cdot\vec{V} = 0$$

Consider the vector identity,

$$\nabla\cdot(\phi\vec{A}) = \vec{A}\cdot\nabla\phi + \phi(\nabla\cdot\vec{A})$$

$\Rightarrow$ $\phi \equiv \rho$, $\vec{A} \equiv \vec{V}$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\vec{V} = 0$$

This is exactly G.1', the conservative form of continuity eqn.

Important observations

1) An incompressible flow model is used to describe flows having the following feature:

"Density changes for individual fluid particles and their effects are negligible."

Note: Different fluid particles may have different densities

e.g. Motion of coffee being mixed with milk

Motion of water in sea and oceans

Smoke rising from a lighted cigarette

Mathematically $\Rightarrow$ $\frac{D\rho}{Dt} = 0$

Now G.1 $\Rightarrow$ $\boxed{\nabla\cdot\vec{V} = 0}$ $\Rightarrow$ Velocity field is divergence free

An incompressible flow velocity field must satisfy the divergence - free criterion. This is true whether the flow is steady or unsteady.

② Steady flow

A flow is steady if any flow variable or property is not a function of time. (It may vary in space).

Using 6.1' $\Rightarrow \cancel{\frac{\partial \rho}{\partial t}}^{0} + \nabla \cdot \rho\vec{V} = 0$

$$\boxed{\nabla \cdot \rho\vec{V} = 0}$$

" $\rho\vec{V}$ must be divergence free "

③ ~~Homo~~ Homogeneous flow

A flow is said to be homogeneous if density variations across different fluid particles at a given instant can be neglected.

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \vec{V}) = 0$$

At an instant

Now, if we further have an incompressible flow {Velocity field is steady or unsteady}

$$\frac{\partial \rho}{\partial t} + 0 = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} = 0} \Rightarrow$$ At all points in flow domain the time variation of density is zero.

Conclusion $\Rightarrow$ If at some time instant the flow is homogeneous and is also incompressible, then the flow remains homogeneous for all subsequent times ($\because \frac{\partial \rho}{\partial t} = 0$).

So, we often encounter incompressible + homogeneous flows in Engineering practice.

Thus, for incompressible, homogeneous flows, $\Rightarrow \boxed{\nabla \cdot \vec{V} = 0, \ \rho = \rho_0} \boxed{\text{(constant)}}$

On the other hand, an incompressible non-homogeneous / stratified flow is one for which $\nabla \cdot \vec{V} = 0$, but $\rho = \text{func}(\vec{r}, t)$.

∞ ———————— ∞.

Governing Equations: Newtons II Law of motion

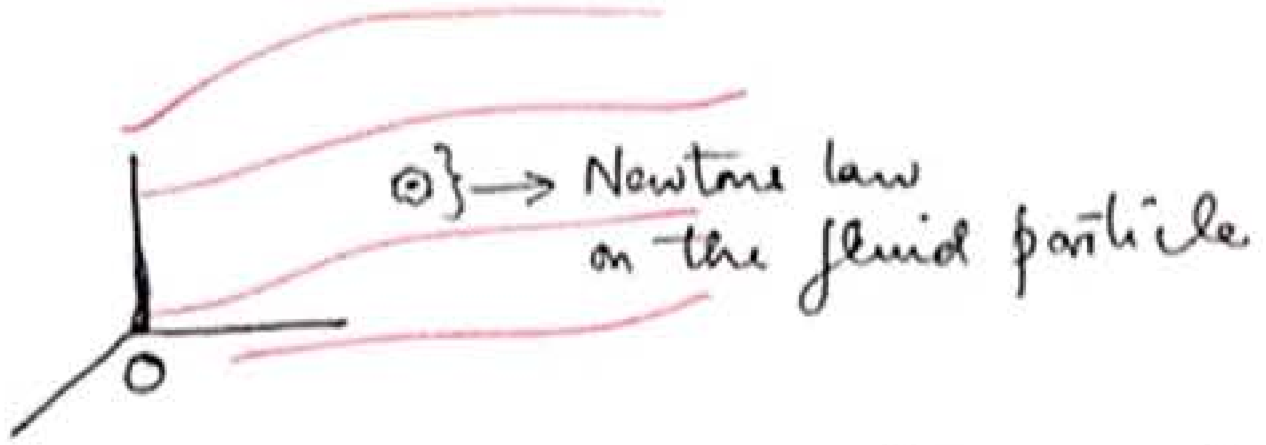
Learning Objectives

Linear Momentum Equation for a fluid particle

Constitutive laws for fluids-Newtonian fluids

Mathematical model for Viscous flows

Newtons II law : The momentum Equation



O : Inertial observer / frame of reference

$$\left\{ (\text{mass})(\text{accn}) \right\}_{\substack{\text{fluid}\\ \text{particle}}} = \left(\Sigma \vec{F}_{ext} \right)_{\text{fluid particle}}$$

Observe the advantage of particle approach. It allows direct application of the principle.

$$(\rho\, \delta\forall)\left(\frac{D\vec{V}}{Dt}\right) = \vec{f}_{body} + \vec{f}_{surf}$$

$$\rho\, \cancel{\delta\forall}\, \frac{D\vec{V}}{Dt} = \rho \underbrace{\vec{B}}_{\substack{\text{force intensity}\\ /\text{mass}}} \cancel{\delta\forall} + (\nabla \cdot \overline{\overline{\sigma}})\cancel{\delta\forall}$$

$$\boxed{\rho \frac{D\vec{V}}{Dt} = \rho \vec{B} + \nabla \cdot \overline{\overline{\sigma}}} \longrightarrow \text{Cauchy's Eqn of motion}$$

$$\boxed{\rho \left(\frac{D\vec{V}}{Dt}\right)_{\text{Inertial}} = \rho \vec{B}_m + \nabla \cdot \overline{\overline{\sigma}}} \text{ — G. 2}$$

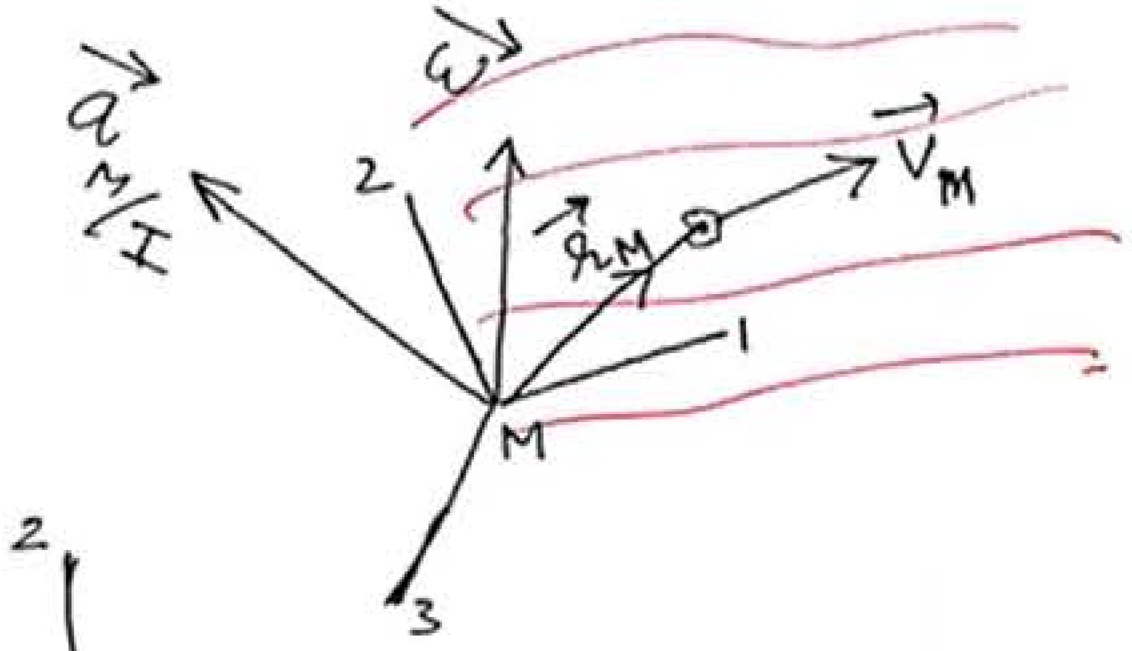
Remarks :

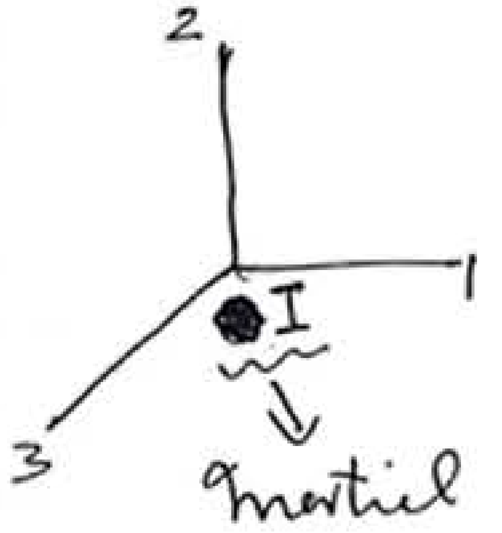
(i) Cauchy's Equation of motion applies to any fluid particle at any point in flow domain and at any ~~point~~ instant of time.

(ii) Involves, density ρ, Velocity $\vec{V}$, and $\overline{\overline{\sigma}}$ as the flow variables.

(iii) This is a vector equation (like any momentum eqn) and has in general 3 scalar components

(iv) For non-inertial moving frames,





$$\left(\frac{D\vec{V}}{Dt}\right)_I = \vec{a}_I = \vec{a}_{M/I} + \frac{d\vec{\omega}}{dt}\times\vec{r}_M + 2(\vec{\omega}\times\vec{V}_M) + \vec{\omega}\times\vec{\omega}\times\vec{r}_M + \underbrace{\vec{a}_M}_{\Downarrow \left(\frac{D\vec{V}}{Dt}\right)_M}$$

$$\left(\frac{D\vec{V}}{Dt}\right)_I = \left(\frac{D\vec{V}}{Dt}\right)_M + \underbrace{\vec{a}_{M/I} + \frac{d\vec{\omega}}{dt}\times\vec{r}_M + 2(\vec{\omega}\times\vec{V}_M) + \vec{\omega}\times\vec{\omega}\times\vec{r}_M}_{\text{4 Extra terms}}$$

$\rightarrow$ Extra acceleration terms

Usage of non-inertial rotating frames

- Fluid flow through rotating components of machines like turbines, compressors.
- Atmospheric & Oceanic flows influenced by earth's rotation
- Fluid sloshing in accelerating containers

x ——— x

Summary :

1. The ~~isothermal~~ flows are governed by basic laws of classical physics like mass consv, momentum eqn, (*) first law of thermodynamics, ...

(*) For flows with heat transfer we require first law of thermodynamics

2. The Physical laws can be applied directly on a fluid particle to obtain equation in non-conservative form.

3. For incompressible, homogeneous flows ($\nabla\cdot\vec{V}=0$, $\rho=\rho_0$), the governing equations $\boxed{G.1 \; \& \; G.2}$ involve only $\underbrace{\vec{V}}_{\text{3 components}}, \underbrace{\bar{\bar{\sigma}}}_{\text{6 components}}$ as flow variables.

A total of nine quantities.

G.1 ⟶ 01
+ scalar. +
G.2 ⟶ 03 = 04 Equations
vector equations

Even for flows without heat transfer the no: of flow variables = 09

G.1 + G.2 = 04 Equations

⟹ G.1 + G.2 ⟹ partially complete model.

Constitutive Relations : Stress and Strain rate tensor relations

Constitutive relative relations provide the missing equations to develop the complete mathematical model for viscous flows.

A. Stress state in absence of viscosity or its effects

We have already learned in our course on Fluid Mechanics that in a state of rest or of uniform motion, the stress tensor at a point is isotropic.

$$\bar{\bar{\sigma}}^{(iso)} = \sigma\,\delta_{ij}\,\hat{e}_i\hat{e}_j = \sigma\,\bar{\bar{I}}$$

$$= \sigma\,\hat{e}_1\hat{e}_1 + \sigma\,\hat{e}_2\hat{e}_2 + \sigma\,\hat{e}_3\hat{e}_3$$

Further the thermodynamic pressure p and σ are related as

$$\sigma = -p$$

$$\boxed{\bar{\bar{\sigma}}^{(iso)} = -p\,\bar{\bar{I}}}$$

B. Constitutive relations (viscous effects included).

It is reasonable to argue that the viscous effects are superimposed on the isotropic stress tensor as additional viscous stress tensor.

$$\bar{\bar{\sigma}} = \bar{\bar{\sigma}}^{iso} + \underbrace{\bar{\bar{\sigma}}^{(v)}}_{\text{viscous stress tensor}}$$

~~[illegible]~~ fluids

- Viscous $\rightarrow$ $\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{S}}, t)$
- Visco-elastic $\rightarrow$ $\bar{\bar{\sigma}}^{(v)} = \text{func}(\text{strain, strain rate, time})$

Viscous fluids
- $\xrightarrow{\text{Time dependant}}$ $\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{S}}, t)$
- $\xrightarrow{\text{Time independant}}$ $\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{S}})$
  - Newtonian
  - Non-Newtonian

Newtonian fluids

$$\bar{\bar{\sigma}}^{(v)} = \text{func}(\bar{\bar{S}}) \longrightarrow \text{linear}$$

$$\left.\sigma_{ij}^{(v)} = \underbrace{\lambda}_{\text{Second coeff. of viscosity}} \epsilon_v \delta_{ij} + 2\underbrace{\mu}_{1^{st}\text{ Coeff. of viscosity}} S_{ij} \right\} \rightarrow \text{Stokes relations}$$

$$\sigma_{11}^{(v)} = 2\mu S_{11} + \lambda\epsilon_v, \quad \sigma_{22}^{(v)} = 2\mu S_{22} + \lambda\epsilon_v, \quad \sigma_{33}^{v} = 2\mu S_{33} + \lambda\epsilon_v$$

$$\sigma_{12}^{(v)} = 2\mu S_{12}$$

$$\sigma_{13}^{(v)} = 2\mu S_{13}$$

$$\sigma_{23}^{(v)} = 2\mu S_{23}$$

Notice that the relations preserve the symmetry of the ~~[illegible]~~ viscous stress tensor

Hence total stress tensor for a Newtonian fluid can be finally expressed as,

$$\boxed{\bar{\bar{\sigma}} = (\underbrace{-p}_{\text{thermodynamic pressure}} + \lambda \underbrace{\epsilon_v}_{\text{vol. strain rate}})\bar{\bar{I}} + 2\mu \bar{\bar{S}}} \longrightarrow G.3$$

Thus, the above stress - strain rate relation for a Newtonian fluid can be regarded as the most general form of constitutive relations for a Newtonian-fluid.

We can write, $\epsilon_v = \sum_{k=1}^{3} S_{kk} = S_{kk}$

$$\boxed{\bar{\bar{\sigma}} = (-p + \lambda S_{kk})\bar{\bar{I}} + 2\mu \bar{\bar{S}}} \longrightarrow G.3'$$

Clearly for incompressible flow,

$$\bar{\bar{\sigma}} = -p\,\bar{\bar{I}} + 2\underbrace{\mu}_{\text{only one coeff. of viscosity}} \bar{\bar{S}} \longrightarrow \text{No role of } \lambda$$

For a general viscous flow of a Newtonian fluid, the isotropic stress is now modified by the $\lambda \epsilon_v$ term so that

$$\bar{\bar{\sigma}}^{(v)iso} = (-p + \lambda \epsilon_v)\bar{\bar{I}}$$
$$= (-p + \lambda S_{kk})\bar{\bar{I}}$$

Interestingly for incompressible flow

$$\bar{\bar{\sigma}}^{(iso)} = -p\,\bar{\bar{I}}$$

Stokes hypothesis on 'λ'

$$\underbrace{p_m}_{\text{Mechanical pressure}} = -\frac{\sigma_{ii}}{3} = -\frac{(\sigma_{11} + \sigma_{22} + \sigma_{33})}{3}$$

Substitute from G.3'

$$p_m = -\frac{1}{3}\left[3(-p + \lambda S_{kk}) + 2\mu S_{kk}\right]$$
$$= -\frac{1}{3}\left[-3p + (3\lambda + 2\mu) S_{kk}\right]$$

$$\boxed{p_m = p - (3\lambda + 2\mu)\frac{S_{kk}}{3}}$$

Stokes assumed, $\boxed{3\lambda + 2\mu = 0}$

$$\lambda = -\frac{2}{3}\mu$$

$$\Rightarrow \boxed{p_m = p}$$

In real experiments, one finds $\epsilon_v \approx 0$ (very small) for liquids.

Even for gases undergoing 'compressible' flows, the difference between p_m and p is not significant except inside shock waves.

Hence for analysis, $p_m = p$

or $\boxed{\lambda = -\frac{2}{3}\mu}$ is widely used.

~~used~~ for all fluid flows.

Stokes hypothesis permits us to work with a single material property $\longrightarrow \mu$: Coeff of viscosity or dynamic viscosity for Newtonian fluids.

Finally, After introducing the Constitutive relations G.3 or G.3',

let us examine, ~~wrt~~ the balance of no: of flow variables Vs no: of equations G.1, G.2, G.3'

Flow variables occurring in G.1, G.2 and G.3

$$= \underset{(01)}{\rho} + \underset{(03)}{\vec{V}} + \underset{(06)}{\bar{\bar{\sigma}}^{(v)}} + \underset{(01)}{p} + \underset{(01)}{\mu} + \underset{(01)}{T}$$

Total no: of flow variables in a general viscous flow = 13

μ is a property of fluid but depends on local temp and pressure of flow $\Rightarrow \mu = \text{func}(\underset{\text{strong}}{T}, \underset{\text{weak}}{p})$

strong weak dependance

The no: of equations

$$= \underbrace{G.1}_{(01)} + \underbrace{G.2}_{(03)} + \underbrace{G.3}_{(06)} + \overset{(01)}{\underbrace{G.4}_{\substack{\text{Energy} \\ \text{or} \\ \text{I}^{st} \text{ law of thermodynamics}}}} + \overset{(02)}{\underbrace{\text{Eqns of state}}_{\substack{\mu = \mu(T, \\ \rho = f(T,}}}}$$

$= 13$!

The Energy equation throws up two more unknowns → thermal conductivity 'K'
→ sp. heat at const. vol. 'C_v'.

This takes variables tally = 15

No: of equations = $\underbrace{13}$ + $\underbrace{02}$

11 physical laws
+
02 Eqn of state

additional equations of state are needed.

$K = f_1(T, p)$

$C_v = f_2(T, p)$

"Thus the mathematical model of a general viscous flow would involve 15 variables and therefore 15 equations (11 physical laws, 4 Eqns of state)."

The most commonly employed, viscous flow model → 'Incompressible, homogeneous, constant property model'

Incomp. → $\nabla \cdot \vec{V} = 0$
homogeneous → $\rho = \rho_0$

Constant property → $\mu = \mu_0$, $K = K_0$, $C_v = C_{v_0}$

Navier-Stokes Equations

Learning Objective

Obtaining the Navier-Stokes equations in primitive variables (Velocity, pressure) for viscous, incompressible, homogenous, constant property flows of a Newtonian fluid

The Navier-Stokes equations

- Continuity (G.1 / G.1')
- Linear Momentum $\longrightarrow$ Cauchy's Eqn. (G.2)
- Constitutive relations $\longrightarrow$ Newtonian fluids (G.3 / G.3')

Consider the simplest model for viscous flows $\longrightarrow$ Incompressible, homogeneous constant property flows

G.1 $\longrightarrow$ $\nabla\cdot\vec{V} = 0$ ($\because$ Incompressible)

G.2 $\longrightarrow$ $\underset{\sim}{\rho_0}\dfrac{D\vec{V}}{Dt} = \underset{\sim}{\rho_0}\vec{B}_m + \nabla\cdot\bar{\bar{\sigma}}$ ($\because$ homogeneous)

G.3$^{\bullet}$ $\longrightarrow$ $\bar{\bar{\sigma}} = -p\bar{\bar{I}} + 2\underset{\sim}{\mu_0}\bar{\bar{S}}$ ($\because$ Constant viscosity)

In obtaining the Navier-stokes equations, substitute the $\bar{\bar{\sigma}}$ tensor from G.3$^{\bullet}$ into G.2

$$\Rightarrow \nabla\cdot\bar{\bar{\sigma}} = \nabla\cdot(-p\bar{\bar{I}}) + 2\mu_0\nabla\cdot\bar{\bar{S}}$$

Let us examine the two terms further.

$$\nabla\cdot(-p\bar{\bar{I}}) = \frac{\hat{e}_i\,\partial}{h_i\,\partial x_i}\cdot\left(-p\,\delta_{jk}\,\hat{e}_j\hat{e}_k\right)$$

$$= \frac{\hat{e}_i\,\partial}{h_i\partial x_i}\cdot\left(-p\,\hat{e}_j\hat{e}_j\right)$$

$$= \hat{e}_i\cdot\left[-\frac{\partial p}{h_i\partial x_i}\hat{e}_j\hat{e}_j - p\frac{\partial\hat{e}_j}{h_i\partial x_i}\hat{e}_j - p\hat{e}_j\frac{\partial\hat{e}_j}{h_i\partial x_i}\right]$$

$$= -\left\{\frac{\partial p}{h_i\partial x_i}(\hat{e}_i\cdot\hat{e}_j)\hat{e}_j + p\left(\hat{e}_i\cdot\frac{\partial\hat{e}_j}{h_i\partial x_i}\right)\hat{e}_j + p(\hat{e}_i\cdot\hat{e}_j)\frac{\partial\hat{e}_j}{h_i\partial x_i}\right\}$$

For orthogonal systems,

$$\hat{e}_i\cdot\hat{e}_j = \delta_{ij}$$

$$= -\left\{\frac{\partial p}{h_i\partial x_i}\delta_{ij}\hat{e}_j + p\,\delta_{ij}\frac{\partial\hat{e}_j}{h_i\partial x_i} + p\left(\hat{e}_i\cdot\frac{\partial\hat{e}_j}{h_i\partial x_i}\right)\hat{e}_j\right\}$$

$$= -\left\{\underbrace{\frac{\partial p}{h_i\partial x_i}\hat{e}_i}_{=\nabla p} + p\frac{\partial\hat{e}_i}{h_i\partial x_i} + p\left(\hat{e}_i\cdot\frac{\partial\hat{e}_j}{h_i\partial x_i}\right)\hat{e}_j\right\}$$

General expression for pressure force/vol in orthogonal systems.

For cartesian: $h_i = 1.0$, $\frac{\partial \hat{e}_i}{\partial x_i} = \frac{\partial \hat{e}_j}{\partial x_i} = 0$

$$\nabla\cdot(-p\bar{\bar{I}}) = -\frac{\partial p}{\partial x_i}\hat{e}_i = -\nabla p$$

For cylindrical

$$\nabla\cdot(-p\bar{\bar{I}}) = -\nabla p - p\frac{\partial \hat{e}_\theta}{r\,\partial\theta} - p\left(\hat{e}_\theta\cdot\frac{\partial \hat{e}_j}{r\,\partial\theta}\right)\hat{e}_j$$

$$= -\nabla p + \cancel{\frac{p}{r}\hat{e}_r} - \cancel{\frac{p}{r}\hat{e}_r}$$

$$\nabla\cdot -p\bar{\bar{I}} = -\nabla p$$

For cartesian, cylindrical

$$\boxed{\nabla\cdot(-p\bar{\bar{I}}) = -\nabla p}$$

The other term involving strain rate tensor can be simplified as,

$$\nabla\cdot\bar{\bar{S}} = \frac{1}{h_i}\frac{\partial S_{ik}}{\partial x_i}\hat{e}_k + S_{jk}\left(\hat{e}_i\cdot\frac{\partial \hat{e}_j}{h_i\,\partial x_i}\right)\hat{e}_k + S_{ik}\frac{\partial \hat{e}_k}{h_i\,\partial x_i}$$

general result for any tensor of rank 2 (Refer to Calculus of tensor sheet)

For Cartesian:

$$\nabla\cdot\bar{\bar{S}} = \frac{\partial S_{ik}}{\partial x_i}\hat{e}_k = \frac{\partial}{\partial x_i}\left[\frac{1}{2}\left(\frac{\partial v_k}{\partial x_i} + \frac{\partial v_i}{\partial x_k}\right)\right]\hat{e}_k$$

$$= \frac{1}{2}\left[\frac{\partial^2 v_k}{\partial x_i\,\partial x_i} + \frac{\partial}{\partial x_k}\underbrace{\left(\frac{\partial v_i}{\partial x_i}\right)}_{\nabla\cdot\vec{V} = 0}\right]\hat{e}_k$$

$$\boxed{\nabla\cdot\bar{\bar{S}} = \frac{1}{2}\left(\frac{\partial^2 v_k}{\partial x_i\,\partial x_i}\right)\hat{e}_k = \frac{1}{2}\nabla^2\vec{V}}$$

For Cylindrical:

$$\nabla\cdot\bar{\bar{S}} = \left[\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta r}}{\partial\theta} + \frac{\partial S_{zr}}{\partial z} + \frac{S_{rr} - S_{\theta\theta}}{r}\right]\hat{e}_r +$$

$$\left[\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial\theta} + \frac{\partial S_{z\theta}}{\partial z} + 2\frac{S_{r\theta}}{r}\right]\hat{e}_\theta +$$

$$\left[\frac{\partial S_{rz}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta z}}{\partial\theta} + \frac{\partial S_{zz}}{\partial z} + \frac{S_{rz}}{r}\right]\hat{e}_z$$

This is directly written using the $\nabla\cdot\bar{\bar{T}}$ expression derived in lecture - 6 (Part-2).

Using the strain rate tensor components in cylindrical coordinates (lecture-7), we can express after a little manipulation

$$\nabla\cdot\overline{\overline{S}} = \left[\frac{1}{2}\nabla^2 v_r - \frac{1}{2}\frac{v_r}{r^2} - \frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta} + \frac{1}{2}\frac{\partial}{\partial r}\cancel{(\nabla\cdot\vec{V})}^{0}\right]\hat{e}_r + \left[\frac{1}{2}\nabla^2 v_\theta - \frac{1}{2}\frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial v_r}{\partial\theta} + \frac{1}{2}\frac{\partial}{r\partial\theta}\cancel{(\nabla\cdot\vec{V})}^{0}\right]\hat{e}_\theta + \left[\frac{1}{2}\nabla^2 v_z + \frac{1}{2}\frac{\partial}{\partial z}\cancel{(\nabla\cdot\vec{V})}^{0}\right]\hat{e}_z$$

$$\nabla\cdot\overline{\overline{S}} = \frac{1}{2}\nabla^2\vec{V}$$

($\because$ This can be verified by applying Laplacian operator ∇^2 in cylindrical coords. on vector $\vec{V}$. Be careful with unit vectors $\hat{e}_r$, $\hat{e}_\theta$).

The effort is rewarded in the sense that $\nabla\cdot\overline{\overline{S}} = \frac{1}{2}\nabla^2\vec{V}$ → Cartesian, → Cylindrical.

Therefore finally, the linear momentum or G.2 eqn can be expressed for incompressible, homogeneous, constant property flow of a Newtonian fluid as,

$$\rho_0\frac{D\vec{V}}{Dt} = \rho_0\vec{B}_m \underbrace{- \nabla p + \mu_0\nabla^2\vec{V}}_{\nabla\cdot\overline{\overline{\sigma}}}$$

The above equation holds for Cartesian as well as cylindrical coordinates.

Finally, we have the following set of equations in primitive variables (velocity, pressure) that govern the viscous, incompressible, homogeneous constant property flow of a Newtonian fluid:

$$\boxed{\begin{aligned} &\nabla\cdot\vec{V} = 0 \\ &\rho_0\frac{D\vec{V}}{Dt} = \rho_0\vec{B}_m \underbrace{-\nabla p}_{\text{pressure force/Vol.}} + \overbrace{\mu_0\nabla^2\vec{V}}^{\text{viscous force/Vol}} \end{aligned}}$$

These are the Navier-Stokes equations

These equations form a complete model as far as no: of variables and no: of equations are concerned.

Variables $\equiv (\vec{V}, p) \longrightarrow 04$

Equations $\equiv (\underbrace{01}_{\text{Continuity}} + \underbrace{03}_{\text{Momentum}}) \longrightarrow 04$

Further, the momentum eqn. reveals that for incomp., homog., const. prop. flow of a Newtonian fluid, the viscous force acting on a fluid particle per unit vol $= \boxed{\mu_0 \nabla^2 \vec{V}.}$

The linear momentum equation reduces to the Euler's Eqn if viscous force is neglected. We learned about Euler's Eqn in our Earlier course on Fluid Mechanics.

Summary

1. The procedure of applying basic laws to a fluid particle directly gives the governing equations for fluid motion

2. For any type of viscous fluid, the basic governing equations (physical laws)
   - Continuity : $\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla\cdot\vec{V} = 0$
   - Linear Momentum : $\rho\frac{D\vec{V}}{Dt} = \rho\vec{B}_m - \nabla p + \nabla\cdot\overline{\overline{\sigma}}^{(v)}$
   - Constitutive : $\overline{\overline{\sigma}}^{(v)} = \text{func}(\overline{\overline{S}}, t)$ or $= \text{func}(\overline{\overline{S}})$.

3. For Newtonian fluids

$$\overline{\overline{\sigma}}^{(v)} = (\lambda S_{kk})\overline{\overline{I}} + 2\mu\overline{\overline{S}}$$

$$\lambda = -\frac{2}{3}\mu$$

4. The Basic physical laws need to be supplemented with equation of states like, $\rightarrow \rho = f(p,T)$, $\mu = f(T,p)$, - - - -

in order to have as many equations as the number of flow variables contained in those equations.

5. The Navier-Stokes equations for viscous, incomp., homogeneous, constant property flow for orthogonal ~~Cartesian~~ coordinate systems (Cartesian, Cylindrical and even spherical) are,

$$\nabla \cdot \vec{V} = 0$$

$$\rho_0 \underbrace{\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V}\right]}_{\frac{D\vec{V}}{Dt}} = \rho_0 \vec{B}_m - \nabla p + \mu_0 \nabla^2 \vec{V}.$$

Mathematical properties of N-S equations and boundary / initial conditions

Learning Objectives

Understanding the mathematical nature of N-S equations (primitive variables) for incompressible, homogenous, constant property flow model

Alternate formulations of the Incompressible, homogenous, constant property model

Mathematical properties of N-S Equations & Boundary Conditions

Incomp. homog. const. prop. flow

$$\rightarrow \quad \nabla \cdot \vec{V} = 0$$

$$\rho_0 \left[\underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{linear}} + \underbrace{(\vec{V} \cdot \nabla)\vec{V}}_{\text{Non-linear term}} \right] = \rho_0 \vec{B}_m - \underbrace{\nabla p}_{\text{linear}} + \underbrace{\mu_0 \nabla^2 \vec{V}}_{\text{linear}}$$

Cartesian $\longrightarrow$ $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

$$\rho_0 \left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right] = \rho_0 B_x - \frac{\partial p}{\partial x} + \mu_0 \overbrace{\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)}^{\nabla^2 u}$$

$$\rho_0 \left[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right] = \rho_0 B_y - \frac{\partial p}{\partial y} + \mu_0 \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

$$\rho_0 \left[\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right] = \rho_0 B_z - \frac{\partial p}{\partial z} + \mu_0 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

Cylindrical $\longrightarrow$ Home work, do it as a self learning exercise

Observations

1. N-S Equation are a set of Partial Differential equations (4 Nos, for 04 unknowns u, v, w, p)

2. They are coupled equations. We do not have explicit equations for u, v, w and p. $\Rightarrow$ solution of each influences the other.

   Equations have to be solved simultaneously or as a system of equations.

3. The momentum equations have a non-linear term $\rightarrow$ convective acceleration.

"Analytical Solutions are extremely limited".

In fact much of the mathematical difficulty, complexity in fluid behaviour arises out of this non-linear character.

4. Highest order of derivatives

- 2^{nd} order in space for velocity components

  $\left(\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 v}{\partial x^2}, \frac{\partial^2 w}{\partial x^2}, \ldots\ldots\ldots \right)$

- 1^{st} order in time for velocity components

  $\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right)$.

- 1^{st} order in space for pressure

Implication on boundary / initial condition requirements.

In obtaining particular solutions to Ordinary differential equation

$\longrightarrow$ Order of equation = Number of conditions required on the variable

~~Therefore~~ A Similar requirement exists for the partial differential equations.

A difference is that order of the equation is associated with each independent variable — space coordinates, time

Thus, we can summarize the order of N-S equations for each ~~variable~~ dependent flow variable u, v, w, p with respect to independent variables (x, y, z, t) and the maximum no: of boundary and initial conditions required for obtaining particular solutions

	space coords.			time
	x	y	z	t
u-vel	Order=2 BC=2	Order=2 BC=2	Order=2 BC=2	Order=1 IC=1
v-vel	"	"	"	"
w-vel	"	"	"	"
p	Order=1 BC=1	Order=1 BC=1	Order=1 BC=1	—

5. Alternate formulations

(i) Conservative body forces:

A body force is conservative if $\nabla \times \vec{B}_m = 0$.

Using vector identity $\nabla \times \nabla \phi = 0$, $\vec{B}_m$ can be expressed as

$$\vec{B}_m = \nabla \phi_B$$

In the N-S equation, the conservative body forces can be combined with pressure forces,

$$\underbrace{\rho_0 \vec{B}_m - \nabla p}_{} = \rho_0 \nabla \phi_B - \nabla p$$

$$= -\nabla \underbrace{\left(p - \rho_0 \phi_B \right)}_{\text{effective or motion pressure } 'p_m' \text{ or } 'p_{eff}'}$$

Consider the example of gravity which is a conservative body force.

$z \uparrow \quad \downarrow g \qquad \vec{B}_m = -g\hat{k}, \quad \phi_B = -gz$

$\therefore$ $\vec{B}_m = \nabla \phi_B$

$$p_m \text{ or } p_{eff} = p - \rho_0 \phi_B = (p + \rho_0 g z)$$

Thus, for flows that are influenced by both gravitational forces and pressure forces, the effects of both ~~B~~ can be combined via a single variable: Effective / Motion pressure. The momentum eqn. for conservative gravity or Body forces becomes:

$$\rho_0 \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} \right] = -\underbrace{\nabla p_m}_{\text{pressure force + body force}} + \mu_0 \nabla^2 \vec{V}$$

Notice that for a fluid at rest,

$\boxed{-\nabla p_m = 0}$ or $\boxed{\nabla p_m = 0}$ $\rightarrow$ at all points

This is the reason for calling it as motion pressure, since its gradient exists only for the case of fluid motion.

This approach of combining conservative body forces with pressure is useful when both forces are involved. When the flow is driven by gravity alone, this approach though valid is not very useful.

(ii) Eliminating "pressure" from N-S Equations : Velocity – Vorticity formulation

$$\rho_0\left[\frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\nabla)\vec{V}\right] = \rho_0\vec{B}_m - \nabla p + \mu_0\nabla^2\vec{V}$$

Consider the following identities in vector calculus,

$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\cdot\nabla\vec{B} + \vec{B}\cdot\nabla\vec{A} + \vec{A}\times\nabla\times\vec{B} + \vec{B}\times\nabla\times\vec{A}$$

$$\nabla\times\nabla\phi = 0$$

$$\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

Using the first identity and third identities we have

$$\nabla(V^2) = 2\big(\vec{V}\cdot\nabla\vec{V} + V\times\underbrace{\nabla\times\vec{V}}_{\vec{\Omega}\,\rightarrow\,\text{Vorticity vector}}\big)$$

or $\nabla(V^2/2) = (\vec{V}\cdot\nabla)\vec{V} + \vec{V}\times\vec{\Omega}$ and

$$\nabla\times\vec{\Omega} = \nabla(\cancelto{0}{\nabla\cdot\vec{V}}) - \nabla^2\vec{V}$$

The momentum eqn can be expressed as,

$$\rho_0\left[\frac{\partial\vec{V}}{\partial t} + \underbrace{\nabla(V^2/2) + \vec{\Omega}\times\vec{V}}_{(\vec{V}\cdot\nabla)\vec{V} = \nabla(V^2/2) - \vec{V}\times\vec{\Omega}}\right] = \rho_0\vec{B}_m - \nabla p - \mu_0(\nabla\times\vec{\Omega})$$

Taking curl on both sides,

$$\rho_0\left[\underbrace{\nabla\times\frac{\partial\vec{V}}{\partial t}}_{\frac{\partial}{\partial t}(\nabla\times\vec{V}) = \frac{\partial\vec{\Omega}}{\partial t}} + \cancelto{0}{\nabla\times\nabla(V^2/2)} + \nabla\times\vec{\Omega}\times\vec{V}\right] = \rho_0\nabla\times\vec{B}_m - \cancelto{0}{\nabla\times\nabla p} - \mu_0(\nabla\times\nabla\times\vec{\Omega})$$

$\because \nabla\times\nabla\phi = 0$

Now

$$\nabla\times\vec{\Omega}\times\vec{V} = (\cancelto{0}{\nabla\cdot\vec{V}})\vec{\Omega} - (\cancelto{0}{\nabla\cdot\vec{\Omega}})\vec{V} + (\vec{V}\cdot\nabla)\vec{\Omega} - (\vec{\Omega}\cdot\nabla)\vec{V}$$

$$\nabla\times\nabla\times\vec{\Omega} = \nabla(\cancelto{0}{\nabla\cdot\vec{\Omega}}) - \nabla^2\vec{\Omega}$$

Use identities
$\nabla\times\vec{A}\times\vec{B} = \ldots$
$\nabla\times\nabla\times\vec{A} = \ldots$

We get,

$$\rho_0\left[\underbrace{\frac{\partial\vec{\Omega}}{\partial t} + (\vec{V}\cdot\nabla)\vec{\Omega}}\right] = \rho_0(\vec{\Omega}\cdot\nabla)\vec{V} + \rho_0(\nabla\times\vec{B}_m) + \mu_0\nabla^2\vec{\Omega}$$

or Vorticity Transport Eqn

$$\boxed{\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega}\cdot\nabla)\vec{V} + \nabla\times\vec{B}_m + \left(\frac{\mu_0}{\rho_0}\right)\nabla^2\vec{\Omega}}$$

The incomp. condition can also be used to obtain a relation between velocity and vorticity.

We know, $\vec{\Omega} = \nabla \times \vec{V}$

or $\nabla \times \vec{\Omega} = \nabla \times \nabla \times \vec{V}$

$= \nabla(\nabla \cdot \vec{V}) - \nabla^2 \vec{V}$

$$\boxed{\nabla^2 \vec{V} = -\nabla \times \vec{\Omega}} \quad \text{—} \; (1)$$

This equation along with Vorticity Transport Equation constitute a Velocity-Vorticity formulation of an incomp., homog., constant property flow.

Note:

i) For 2D flows, $\vec{\Omega} \perp \nabla \vec{V}$

$\Rightarrow \vec{\Omega} \cdot \nabla \vec{V} = 0$

$\Rightarrow$ V. T. E $\longrightarrow$ $\dfrac{D\vec{\Omega}}{Dt} = \nabla \times \vec{B}_m + \left(\dfrac{\mu_0}{\rho_0}\right) \nabla^2 \vec{\Omega}$

(Vorticity Transport Eqn)

ii) For conservative body forces

$\nabla \times \vec{B}_m = 0$

V.T.E $\Rightarrow \dfrac{D\vec{\Omega}}{Dt} = (\vec{\Omega} \cdot \nabla)\vec{V} + \left(\dfrac{\mu_0}{\rho_0}\right) \nabla^2 \vec{\Omega}$ } Consv. Body forces do not generate vorticity.

Boundary Conditions on a material interface

Learning Objectives

Types of Boundary conditions on a Solid-Fluid and a Fluid-Fluid material interface

Understanding the role of the boundary conditions in influencing / generating fluid motion

Boundary Conditions

As shown in the previous lecture (No. 12) the particular flow problem solutions using N-S equations require appropriate number of Boundary Conditions on the various flow variables.

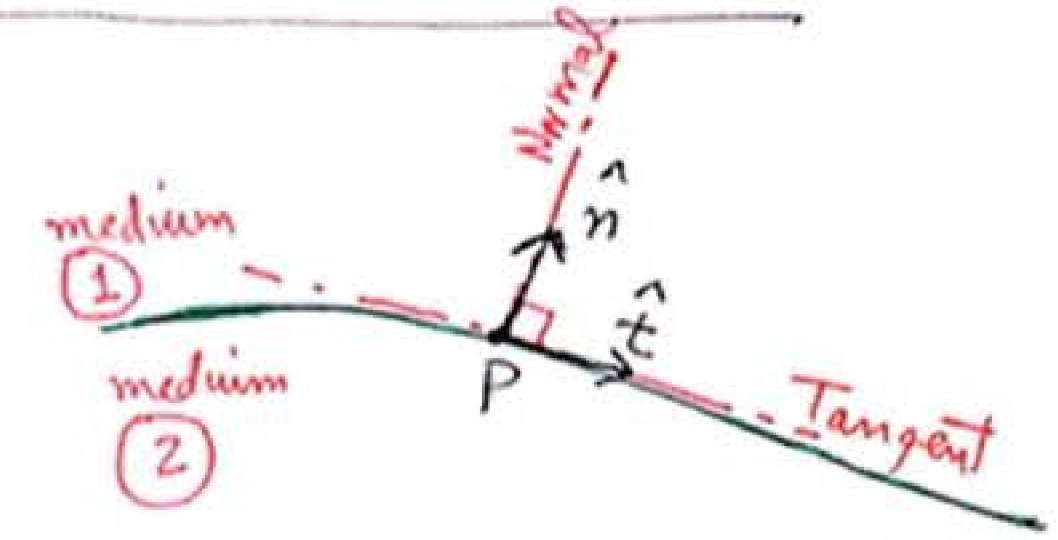
"The flow behaviour of the same fluid in different geometries is different because of different boundary / initial conditions"

The conditions at a material boundary or interface (solid-fluid interface, fluid-fluid interface) can be broadly classified as:

A. Kinematic conditions

B. Dynamic conditions

Kinematic Conditions



~~The normal velocity varies in a~~

At any pt (say P) on the instantaneous interface between two continuous media, the following two conditions apply:

1] The local normal velocity is ~~continuous~~ continuous across the interface $\longrightarrow$ **No penetration condition**

Mathematically

$$(\vec{V})_1 \cdot \hat{n} = (\vec{V}_I \cdot \hat{n}) = (\vec{V})_2 \cdot \hat{n} \quad \text{--- (B.1)}$$

$\vec{V}_I$: velocity at interface

$(\vec{V})_1$: velocity in the neighborhood of interface on medium ① side

$(\vec{V})_2$: Velocity in the neighborhood of the interface on medium ② side.

Physically the condition implies that particles ~~medium~~ of media on either side do not cross the interface.

2] The local tangential component of velocity is continuous across the interface ⟶ $\boxed{\text{No-slip condition}}$

Mathematically

$$(\vec{V})_1 \cdot \hat{t} = (\vec{V})_I \cdot \hat{t} = (\vec{V})_2 \cdot \hat{t} \quad \text{—— (B·2)}$$

Note : In 3D, a surface/interface in general can be characterized locally by one normal and two tangential direction.

Thus in 3D ⟶ At any pt on the interface we can write one no-penetration condition and two no-slip conditions.

Limitations :

1. The No-penetration condition is generally applicable to a (Solid - Fluid) interface which is non-porous. (gas liquid)

This condition must be used with care when a (liquid - liquid) interface or a (liquid - gaseous) interface is involved.

A liquid - liquid interface can only exist for $\boxed{\text{immiscible pair of liquids}}$. For such cases No-penetration can be used.

For a (liquid - gaseous) interface, the no-penetration condition can be used only if the $\boxed{\text{evaporation effect is negligible}}$.

2. The no-slip condition is applicable to any type of $\underset{\text{(solid -fluid)}}{\text{S-F}}$ or $\underset{\text{(fluid -fluid)}}{\text{F-F}}$ interface.

An important Remark

The No-penetration and No-slip conditions are two independent conditions.

However, the two conditions when applicable to a S-F or F-F interface can be combined as follows:

$$(\vec{V})_1 \cdot \hat{n} = (\vec{V})_I \cdot \hat{n} = (\vec{V})_2 \cdot \hat{n}$$

$$\Rightarrow \boxed{\vec{V}_1 \cdot \hat{n} = \vec{V}_2 \cdot \hat{n}}$$

$$(\vec{V})_1 \cdot \hat{t} = (\vec{V})_I \cdot \hat{t} = (\vec{V})_2 \cdot \hat{t}$$

$$\boxed{(\vec{V})_1 \cdot \hat{t} = (\vec{V})_2 \cdot \hat{t}}$$

Since normal as well as tangential components are equal on either side of interface

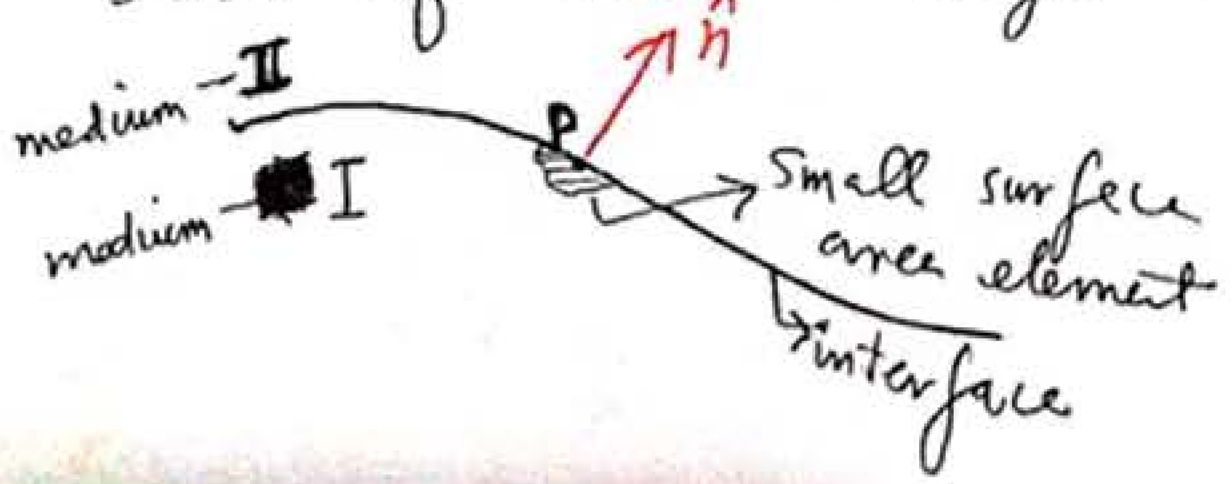
$$\Rightarrow \boxed{\vec{V}_1 = \vec{V}_2} \rightarrow (B\cdot1) + (B\cdot2)$$

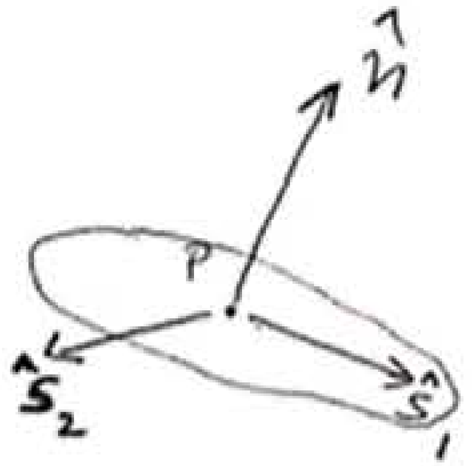
In some texts the last condition $(\vec{V})_1 = (\vec{V})_2$ is incorrectly mentioned as no-slip condition

These kinematic conditions (B·1) & (B·2) are valid ~~under~~ in the continuum regime. i.e the media on either side of the interface $<$ S-F, F-F can be treated as a continuum.

Dynamic Conditions

The dynamic conditions are relations between stress vector components in the neighborhood of a point lying on the S-F or F-F interface, on either side of the interface.





$\hat{s}_1$, $\hat{s}_2$ are any two mutually orthogonal tangential unit vectors at point P on the elemental surface ~~elem~~ area ds.

The dynamic conditions can be expressed as:

a) <u>Solid - fluid</u> surface

$$(\sigma_{nn})_I = (\sigma_{nn})_{II}$$

$$\cancel{(\sigma_{ns_1})_I = (\sigma_{ns_2})} \quad (\sigma_{ns_1})_I = (\sigma_{ns_1})_{II}$$

$$(\sigma_{ns_2})_I = (\sigma_{ns_2})_{II}$$

"All local stress vector components are continuous across the surface"

b) <u>For a Fluid - Fluid interface</u>

surface tension — Radii of curvature

$$(\sigma_{nn})_I - (\sigma_{nn})_{II} = S\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$(\sigma_{ns_1})_I - (\sigma_{ns_1})_{II} = -\frac{\partial S}{\partial s_1}, \quad (\sigma_{ns_2})_I - (\sigma_{ns_2})_{II} = -\frac{\partial S}{\partial s_2}$$

* "All local stress vector components are discontinuous across the surface".

For a planar fluid - fluid interface

$\Rightarrow R_1 = R_2 \to \infty$

$\Rightarrow (\sigma_{nn})_I = (\sigma_{nn})_{II} \Rightarrow$ Normal stresses are continuous

If surface tension is negligible / or does not vary along the interface

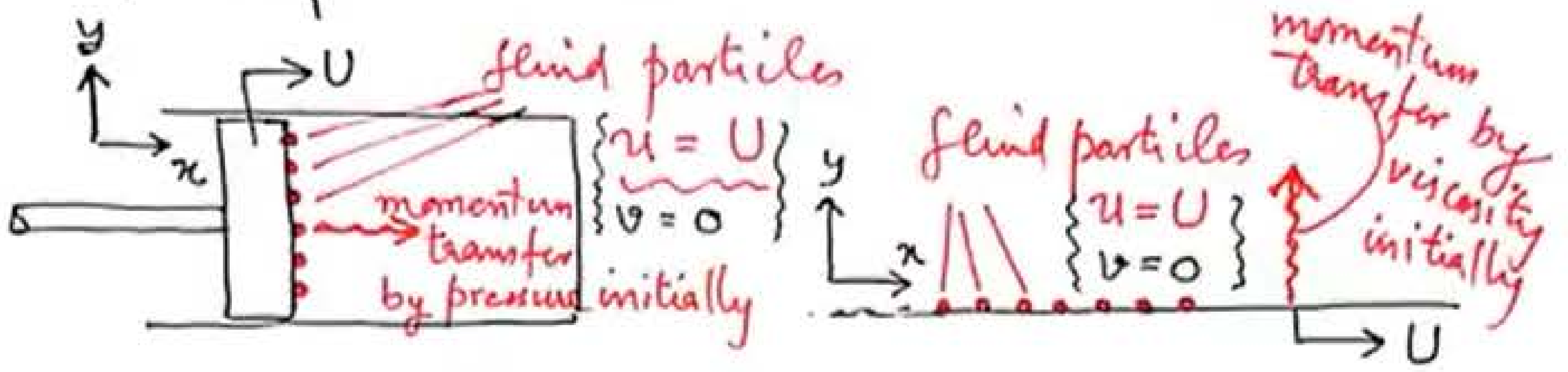
$$\Rightarrow \left.\begin{array}{l}(\sigma_{ns_1})_I = (\sigma_{ns_1})_{II} \\ (\sigma_{ns_2})_I = (\sigma_{ns_2})_{II}\end{array}\right\} \Rightarrow \text{Shear stresses are also continuous}$$

<u>Summary</u>:

1. The presence of a physical interface of a fluid with a <u>solid</u> or <u>another fluid</u> is communicated through the kinematic & Dynamic boundary conditions
2. The fluid motion is influenced by these boundary conditions.
3. For a ~~r~~ <u>rigid</u> <u>solid - fluid</u> interface <u>only the kinematic boundary conditions are relevant</u>.

4. Generation of fluid motion

The kinematic boundary condition can play a very important role in generation of fluid motion. look at some examples below:



fluid motion caused by no-penetration condition

fluid motion caused by no-slip condition

Note: The no-slip condition transfers fluid motion to interior fluid particles by viscous effects. $\Rightarrow$ No-slip condition is able to influence the fluid motion only through viscosity

This implies that for inviscid flow analysis or models, the no-slip condition is not relevant.

Dimensionless Formulation

Learning Objectives

Introduction to the concept of dimensionless variables

Conversion of N-S equations to dimensionless forms

Dimensionless numbers / parameters in N-S equations

Dimensionless Formulation

The Navier–Stokes Equations derived earlier for an incomp., homogeneous, const property flow contain dimensional quantities (t, x, y, z or any space coords), {~~[illegible]~~ ($\vec{V}$, p)} and some dimensional parameters like (ρ_0, μ_0). Therefore we have a relation(s) between several dimensional variables.

"Invoking Buckingham - π - theorem, it is possible to reduce the parameters by combining them into dimensionless group(s) or parameters".

The approach adopted is slightly different to what was taught earlier in MEC 2310.

This is because the viscous flow model (Governing eqns. + related boundary conditions / initial) was not developed earlier

The basic idea in converting the N-S equations (or any other type of equation) into dimensionless form is to replace all dimensional variables (including space coordinates and time as they are also variables) by corresponding dimensionless variables.

How this is achieved?

$$Q_{Dim} = \underbrace{(\text{scale})}_{\text{order of magnitude expected for } Q_{Dim}} \underbrace{Q^*}_{\text{Dimensionless}} = Q_s Q^*$$

For example, in a group of 60 students in a class, we consider ~~this~~ the "Height of a student" say H as a variable

Then $H = \underbrace{(\text{Scale})}_{\text{order of magnitude}} H^*$

order of magnitude $\approx (1\text{ m} - 1.8\text{ m}) \rightarrow$ Scale $= H_S = 1.5\text{ m}$

Now consider the N-S eqns,

$$\nabla \cdot \vec{V} = 0$$

$$\rho_o \left[\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} \right] = \rho_o \vec{B}_m - \nabla p + \mu_o \nabla^2 \vec{V}$$

The space coordinates are ~~..~~ present in ~~the~~ $\nabla \equiv$ operator $\equiv \hat{e}_i \frac{\partial}{h_i \partial x_i}$

$$\equiv \hat{e}_i \underbrace{\frac{\partial}{\partial s_i}}$$

lengths of displ. vectors along different coordinate directions

Steps involved in converting N-S eqns in dimensionless form

1. Identify the variables — flow vars., space, time — and choose the corresponding scales

$$(\text{space. displ.}) = L_s\,(\text{space-displ.})^*$$

$$\text{time} = t = t_s\, t^*$$

$$\vec{V} = U_s \vec{V}^*, \quad p = p_s p^*$$

Notice that if instead of absolute pressure, we choose to work with gauge pressure $p_g = (p - p_o)$ — p_o: any reference press e.g. p_{atm}

~~the~~ N-S eqn pressure force term remains unaffected i.e. $\nabla p = \nabla p_g$

(since p_o is a const./fixed value).

Now the ~~..~~ gauge pressure can be converted into dimensionless form as

$$p_g = p_s\, p_g^*$$

p_g^* → dimensionless gauge pressure.

The use of gauge pressure is for ease and convenience in Engineering practice. For non-dimensional approach, we can work with abs pressure also.

~~2.~~ Note: The parameters ρ_o, μ_o are left as it is and not converted into dimensionless parameters.

2. Converting the Equation

$$\nabla \equiv \hat{e}_i \frac{\partial}{\partial s_i} \equiv \hat{e}_i \frac{\partial}{L_s \partial s_i^*} \equiv \frac{1}{L_s}\underbrace{\left(\hat{e}_i \frac{\partial}{\partial s_i^*}\right)}_{= \nabla^*}$$

$$\nabla \equiv \frac{1}{L_s} \nabla^*$$

$$\nabla^2 \equiv \nabla \cdot \nabla \equiv \left(\frac{1}{L_s}\right)^2 \nabla^{*2}$$

$$\frac{\partial}{\partial t} \equiv \frac{1}{t_s} \frac{\partial}{\partial t^*}$$

$$\vec{V} = U_s \vec{V}^*, \quad p = p_s p^*$$

→ Continuity: $\nabla \cdot \vec{V} = 0$

$$\left(\frac{1}{L_s} \nabla^*\right) \cdot \left(U_s \vec{V}^*\right) = 0$$

$$\frac{U_s}{L_s}\left(\nabla^* \cdot \vec{V}^*\right) = 0$$

$$\boxed{\nabla^* \cdot \vec{V}^* = 0}$$

→ Momentum Eqn:

$$\rho_o \left[\frac{U_s}{t_s} \frac{\partial \vec{V}^*}{\partial t^*} + \frac{U_s^2}{L_s}\left(\vec{V}^* \cdot \nabla^*\right)\vec{V}^*\right] = \rho_o \underbrace{\vec{g}}_{\vec{B}_m} - \frac{p_s}{L_s} \nabla^* p^* + \frac{\mu_o U_s}{L_s^2} \nabla^{*2} \vec{V}^*$$

$$\rho_o \frac{U_s^2}{L_s}\left[\frac{L_s}{t_s U_s} \frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \nabla\right)\vec{V}^*\right] = \rho_o \vec{g} - \frac{p_s}{L_s} \nabla^* p^* + \frac{\mu_o U_s}{L_s^2} \nabla^{*2} \vec{V}^*$$

Divide by $\rho_o U_s^2 / L_s$

$$\frac{L_s}{t_s U_s} \frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \nabla\right)\vec{V}^* = \frac{L_s}{U_s^2} \vec{g} - \frac{p_s}{\rho_o U_s^2} \nabla^* p^* + \left(\frac{\mu_o}{\rho_o U_s L_s}\right) \nabla^{*2} \vec{V}^*$$

$$\vec{g} = g\,\hat{i}_g$$

$$\boxed{\left(\frac{L_s}{t_s U_s}\right) \frac{\partial \vec{V}^*}{\partial t^*} + \left(\vec{V}^* \cdot \nabla^*\right)\vec{V}^* = \left(\frac{g L_s}{U_s^2}\right) \hat{i}_g - \left(\frac{p_s}{\rho_o U_s^2}\right) \nabla^* p^* + \left(\frac{\mu_o}{\rho_o U_s L_s}\right) \nabla^{*2} \vec{V}^*}$$

Dimensionless parameters in Momentum eqn:

1) $\pi_1 = \dfrac{L_s}{t_s U_s}$ 2) $\dfrac{g L_s}{U_s^2} = \pi_2$

3) $\pi_3 = \dfrac{p_s}{\rho_o U_s^2}$ 4) $\dfrac{\mu_o}{\rho_o U_s L_s} = \pi_4$

π_1 plays a role only when dealing with unsteady flows or performing a transient analysis of a steady flow (time history of how fast the steady state is attained).

Flows are unsteady — Unsteady forcing: generally periodic having period $= T_f$ — Internal instabilities (Turbulent flows).

When the time scale 't_s' cannot be assessed, i) Transient analysis of a steady flow or ii) unsteady flow caused by internal instabilities

$t_s = L_s/U_s \equiv$ Residence time scale $= t_R$

$$\boxed{\pi_1 = \frac{L_s}{t_s U_s} = 1.0} \Rightarrow \pi_1 \text{ is fixed at unity}$$

For unsteady forcing, $\pi_1 = \frac{L_s}{T_f U_s} = \frac{t_R}{T_f}$

$$\pi_2 = \frac{g L_s}{U_s^2} = \frac{g}{U_s^2/L_s} = \frac{\text{gravitational accn}}{\text{order of fluid accn or inertia}}$$

This π-group is in the name of Froude and defined as $Fr = \frac{U_s}{\sqrt{g L_s}}$

$$\Rightarrow \boxed{\pi_2 = \frac{1}{Fr^2}}$$

For flows having very high Fr, π_2 becomes small $\Rightarrow$ gravity terms in momentum eqn $(= \pi_2 \hat{i}_g)$ becomes small and can be neglected.

$\pi_3 : \rightarrow \frac{p_s}{\rho_0 U_s^2}$, for incompressible homogeneous flow, if we apply Bernoulli's theorem along a streamline then changes in pressure have an order $\sim \rho_0 U_s^2$

$\Rightarrow p_s = \rho_0 U_s^2$

$\Rightarrow \boxed{\pi_3 = 1.0} \rightarrow$ becomes fixed!

$$\pi_4 : \frac{\mu_0}{\rho_0 U_s L_s} \rightarrow \pi_4 = \frac{\mu_0 U_s}{L_s^2} \times \frac{1}{\rho_0 U_s^2/L_s}$$

$$= \frac{\text{Order of viscous stresses}}{\text{order of fluid accn or inertia}}$$

This π_4- group or parameter is related to a very celeberated / important dimensionless number for all viscous flows introduced for the first time by Reynolds.

Reynolds no: = Re = $\dfrac{\rho_0 U_s L_s}{\mu_0}$

$$\boxed{\pi_4 = \frac{1}{Re}}$$

~~Firstly~~ As Re becomes small, the flow experiences large effects of viscosity and as Re becomes large, the overall effects of viscosity becomes small.

A word of caution!

For any viscous flow, even if Re is very large (π_4 is very small), the viscous term $\equiv \frac{1}{Re} \nabla^{*2} \vec{V}^{*}$ cannot be dropped entirely, as it is containing the highest order (in space) derivatives of velocity. $\Rightarrow$ BC requirement

Summary

1. Dimensionless formulations directly generate the Dimensionless π-groups or numbers relevant for a given flow model. (Alternative to Buckingham π-theorem)

Buckingham π-theorem is useful if the governing equations themselves are not known.

2. The relative importance of various terms in N-S equations can be judged by the magnitude of the various π- groups or Dimensionless parameters involved

Dynamic Similarity

Learning Objectives

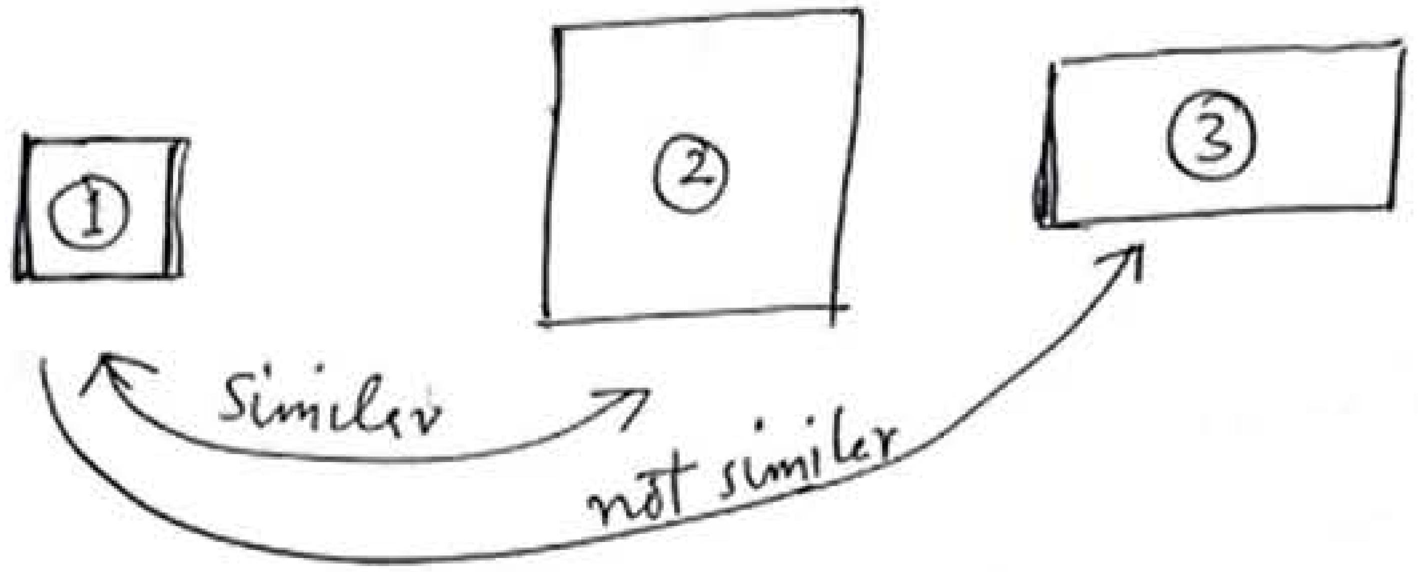
Concepts of geometric and kinematic similarity in flow problems

Dynamic similarity and its relation with dimensionless parameters

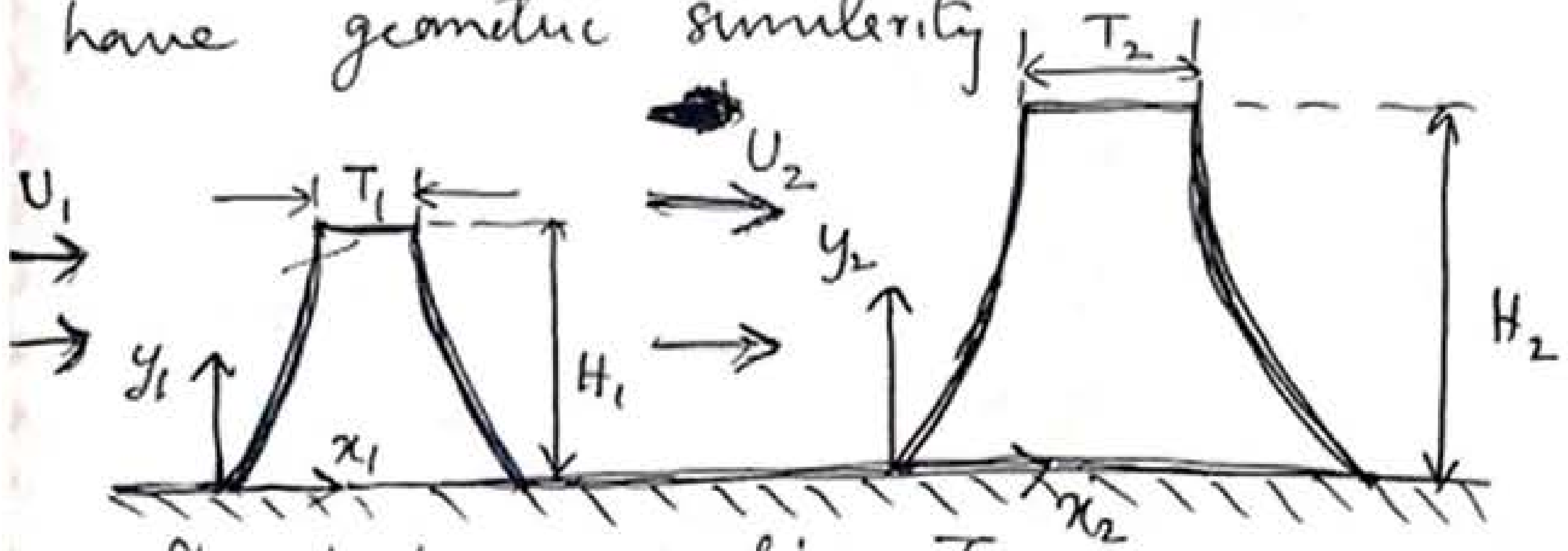
Using the ideas of Dynamic Similarity

Dynamic Similarity

We have the concept of geometric similarity. Two geometries are similar when the ~~different~~ shapes are similar and the corresponding dimensions have the same ratio.

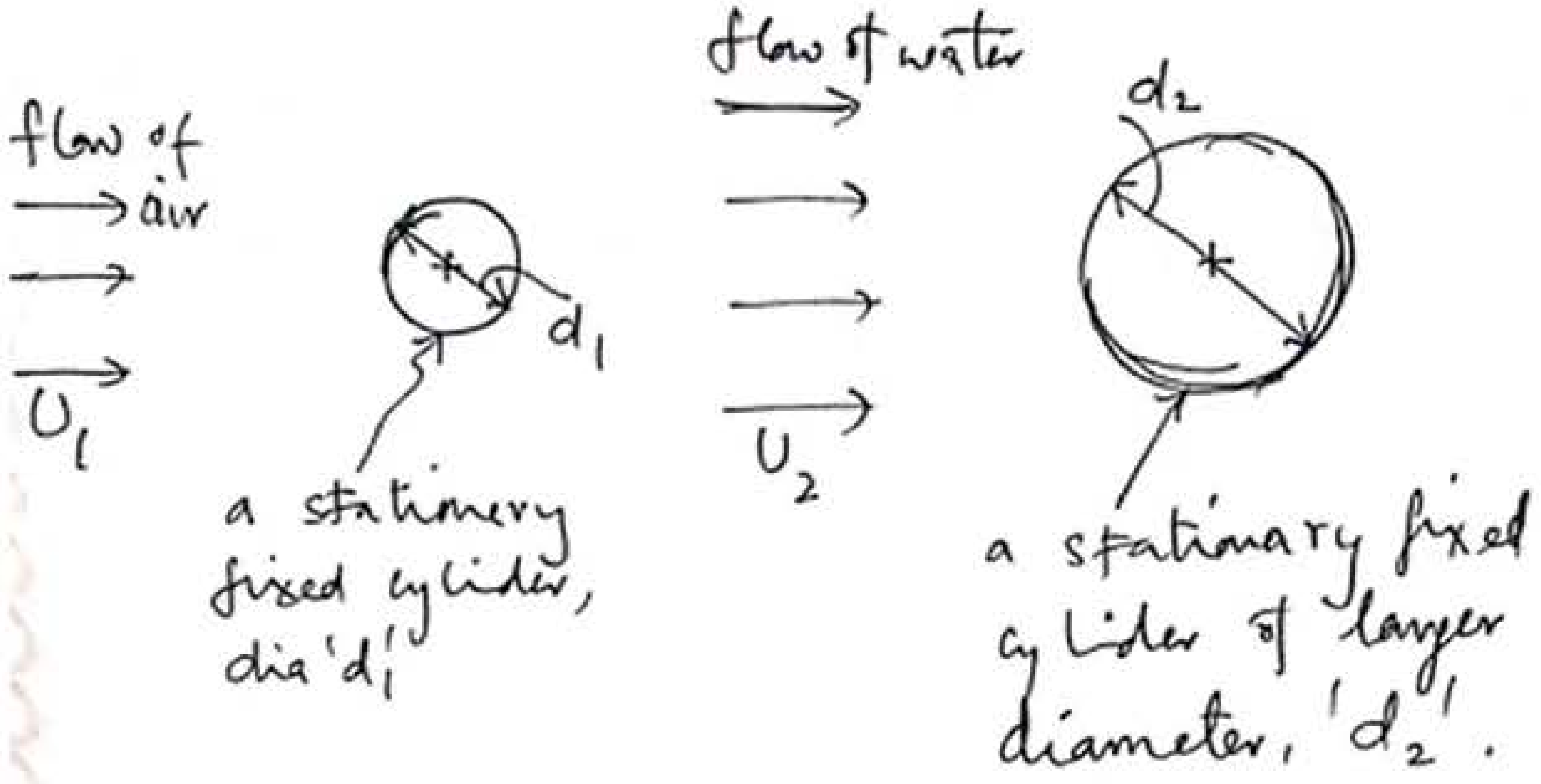


In fluid flow problems also we can have geometric similarity



Steady flow past a cooling Tower

$\frac{T_2}{T_1} = \frac{H_2}{H_1}$, side geometry is described by the same type of function



Steady, Two-Dimensional flow past a circular cylinder

In the above examples apart from geometric similarity, there is similarity of the velocity boundary conditions (No-slip + No-penetration S-F interfaces and same type of free-stream conditions). Such a similarity is termed as kinematic similarity.

When working with unsteady flows, kinematic similarity ~~flows~~ also includes ~~and~~ initial conditions

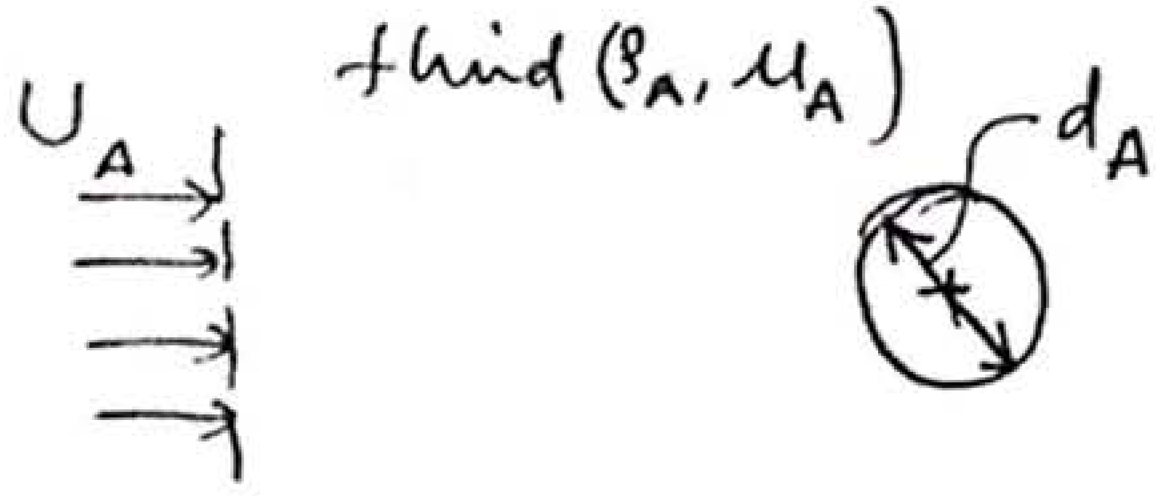
Two flows are said to be "Dynamically similar" when the following conditions hold:

i) Two flows are geometrically and kinematically similar

ii) The two flows have the same dimensionless solution when their solutions (dimensionless) are obtained in the dimensionless domains.

Next we will examine the conditions that can lead to dynamic similarity between two geometrically ad kinematically similar flows.

Consider a incomp., homogeneous, constant property flow of two different fluids around two spherical objects of different size. The objects are immersed in a large expanse of fluids.



Prob. A.

$L_s = d_A$, $U_s = U_A$, $p_s = \rho_A U_A^2$, $t_s = d_A / U_A = t_R$

$\Downarrow$ $(\pi_1 = \pi_3 = 1.0)$

$$\nabla^* \cdot \vec{V}_A^* = 0$$

$$\frac{D\vec{V}_A^*}{Dt^*} = -\nabla^* p_{mA}^* + \frac{1}{Re_A} \nabla^{*2} \vec{V}_A^*$$

$$Re_A = \frac{\rho_A U_A d_A}{\mu_A}$$

Prob. B

$L_s = d_B$, $U_s = U_B$, $p_s = \rho_B U_B^2$, $t_s = d_B / U_B = t_R$

$\Downarrow$ $(\pi_1 = \pi_3 = 1.0)$

$$\nabla^* \cdot \vec{V}_B^* = 0$$

$$\frac{D\vec{V}_B^*}{Dt^*} = -\nabla^* p_{mB}^* + \frac{1}{Re_B} \nabla^{*2} \vec{V}_B^*$$

$$Re_B = \frac{\rho_B U_B d_B}{\mu_B}$$

"Since Re is the only dimensionless parameter for this problem, the dimensionless solutions would depend on Re only"

$\Rightarrow$ If $Re_A = Re_B$.

we can conclude

$$\left.\begin{aligned} \vec{V}_A^*(\vec{r}^*, t^*) &= \vec{V}_B^*(\vec{r}^*, t^*) \\ p_{mA}^*(\vec{r}^*, t^*) &= p_{mB}^*(\vec{r}^*, t^*) \end{aligned}\right\} \text{Dynamic similarity}$$

What has been illustrated by the above example can be summarized as follows:

~~In a flow problem~~

Two geometrically, kinematically similar flows are dynamically similar if $\boxed{\text{all}}$ the dimensionless parameters or π-groups relevant in the dimensionless formulations (Equations + B.C + I.C) of the two problems are made 'identical' (values are kept same).

Finally, we examine the relations between the dimensional solutions of the two problems under the condition of Dynamic similarity.

$$\vec{V}_A^* = \vec{V}_B^* \quad \text{at some } (\vec{r}^*, t^*)$$

$$\Rightarrow \frac{\vec{V}_A(\vec{r}_A, t_A)}{U_A} = \frac{\vec{V}_B(\vec{r}_B, t_B)}{U_B}$$

when $\underbrace{\frac{\vec{r}_A}{d_A} = \frac{\vec{r}_B}{d_B}}_{\text{corresponding pts}}$ & $\cancel{\frac{t_A d_A}{U_A} = \frac{t_B d_B}{U_B}}$

$$\underbrace{\frac{t_A U_A}{d_A} = \frac{t_B U_B}{d_B}}_{\text{corresponding time instants}}$$

Using Dynamic Similarity

Dynamic Similarity paves the way and forms the basis of carrying out scaled up/down experiments of real world flows / flows ~~through~~ in various Engineering applications.

For example, it is possible to test scaled down models of airplane / aircraft components / missiles etc in a suitably designed laboratory experimental apparatus.

Even in carrying out computer simulations of flows, the idea of matching the π-groups or dimensionless parameters to achieve dynamic similarity is exploited.

Summary :

1. Two flows are dynamically similar if :
   a) They possess geometric + kinematic similarity (velocity BC's & IC)
   b) ~~All~~ the dimensionless parameters relevant to the problem for the two cases must match in their respective magnitudes.

2. Under the condition of dynamic similarity, ~~the~~ flowfield of the two flows can be related to each other. Thus if one flow field is known or determined, the other can be predicted using the above relation(s).

3. This (Dynamic Similarity) forms ~~the~~ basis of studying / simulating many real world flows through scaled up / down physical experiments in laboratories and even in virtual experiments on computers.